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TWO METHODS OF INVERTING MATRICES

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1. Introduction. The problem of computing the inverse of a nonsingular matrix has widespread applications and so new methods of attacking this problem need no apology. Here, we shall present two methods of computing A^{-1} where A is a nonsingular $n\times n$ matrix. The first method produces each column (or row) vector of A^{-1}; the second method produces A^{-1} itself. Our first approach utilizes the adjoint of A, i.e., $(A_{ij})^t$ where A_{ij} is the cofactor of (i, j) in A; this is particularly interesting because the adjoint representation of an inverse is thought to possess only theoretical value, not computational value. Our second approach exploits the relationship between A^{-1} and B^{-1} where B is a matrix obtained from A by applying a permitted row-reduction operation. Both methods flow from algebraic properties of a generalized *cross product;* this is a straightforward generalization of the vector (or cross) product of $\mathcal{V}_3(R)$.

2. A generalized cross product. First, we present the usual *cross* product of $\mathcal{V}_3(R)$. Let $\{\alpha, \beta\}\subset R_3$ (i.e., let α and β be ordered triples of real numbers) and let M be a 3×3 matrix such that $M_2=\alpha$ and $M_3=\beta$. (Here M_i denotes the ith row vector of M, whereas ${}_iM$ denotes the ith column vector of M). Then by the *cross product* of α and β we mean the ordered triple whose terms are the cofactors of (1, 1), (1, 2), and (1, 3) in M. This vector is denoted by $\alpha\times\beta$. We now generalize this mapping; by a *t-vector* we mean an ordered t-tuple of scalars.

DEFINITION 2.1. *Let $n>1$, let $(\alpha_2, \cdots, \alpha_n)$ be any ordered $(n-1)$-tuple of n-vectors, and let M be an $n\times n$ matrix such that $M_i=\alpha_i$, $i=2, \cdots, n$. By the cross product of $\alpha_2, \cdots, \alpha_n$ we mean the n-vector whose ith term is the cofactor of $(1, i)$ in M, $i=1, \cdots, n$. This vector is denoted by $\times(\alpha_2, \cdots, \alpha_n)$.*

Notice that $\times$ is a mapping that associates an n-vector with each ordered $(n-1)$-tuple of n-vectors, whenever n is a natural number greater than 1. This is something like the mapping *det*; *det* associates a scalar with each ordered n-tuple of n-vectors, whenever $n\in N$. Moreover, $\times$ and *det* possess similar properties. The following properties of $\times$ flow from corresponding properties of *det.*

LEMMA 2.1. *Let $\alpha_1, \cdots, \alpha_n$ be any $(n+1)$-vectors and let γ be any transposition of $\{1, \cdots, n\}$. Then $\times(\alpha_{\gamma(1)}, \cdots, \alpha_{\gamma(n)})=-\times(\alpha_1, \cdots, \alpha_n)$.*

LEMMA 2.2. *Let $\alpha_1, \cdots, \alpha_n$ be any $(n+1)$-vectors and let k be any scalar. Then $\times(\alpha_1, \cdots, k\alpha_r, \cdots, \alpha_n)=k\times(\alpha_1, \cdots, \alpha_r, \cdots, \alpha_n)$.*

LEMMA 2.3. *Let $\alpha_1, \cdots, \alpha_n$ be any $(n+1)$-vectors and let r and s be distinct members of $\{1, \cdots, n\}$. Then*

$$\times(\alpha_1, \cdots, \underset{r\text{th } term}{\alpha_r + \alpha_s}, \cdots, \alpha_n) = \times(\alpha_1, \cdots, \alpha_r, \cdots, \alpha_n).$$

In view of these results we can simplify the computation of a cross product by first row-reducing the matrix whose row vectors are the vectors involved in the cross product, keeping track of all row interchanges and scalar multiplications. Preferably, we shall avoid row interchanges and shall reverse all scalar multiplications; we can carry this out conveniently by using Lemma 2.4, which follows, in place of the preceding lemmas.

LEMMA 2.4: *Let* $\alpha_1, \cdots, \alpha_n$ *be any* $(n+1)$*-vectors, let* k *be any scalar, and let* r *and* s *be distinct members of* $\{1, \cdots, n\}$. *Then*

$$\times(\alpha_1, \cdots, \underbrace{\alpha_r + k\alpha_s}_{r\text{th } term}, \cdots, \alpha_n) = \times(\alpha_1, \cdots, \alpha_r, \cdots, \alpha_n).$$

Our procedure is to row-reduce by applying Lemma 2.4; of course, the required cross product is found from the reduced matrix by computing cofactors. We now illustrate our technique.

Example 1. Compute $\times((1, -2, 2, 1), (2, -5, 4, 1), (-1, -2, 0, 1))$.

Solution. After several applications of Lemma 2.4 we find that

$$\begin{pmatrix} 1 & -2 & 2 & 1 \\ 2 & -5 & 4 & 1 \\ -1 & -2 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & -3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 2 & 6 \end{pmatrix}.$$

Thus

$$\times((1, -2, 2, 1), (2, -5, 4, 1), (-1, -2, 0, 1))$$
$$= \times((1, 0, 0, -3), (0, -1, 0, -1), (0, 0, 2, 6)) = (6, -2, -6, 2).$$

We mention that $\times(\alpha_1, \cdots, \alpha_n)$ is orthogonal to each of its arguments. Moreover, given that $\alpha_1, \cdots, \alpha_n$ are distinct $(n+1)$-vectors, $\times(\alpha_1, \cdots, \alpha_n) = \mathbf{0}$ iff $\{\alpha_1, \cdots, \alpha_n\}$ is linearly dependent.

3. Computing the column vectors of an inverse. Now that we possess a simple method of computing cross products, we can exploit the classical expression for the inverse of a matrix A in terms of its adjoint, namely

$$A^{-1} = \frac{1}{|A|}(A_{ij})^t \tag{1}$$

where A_{ij} is the cofactor of (i, j) in A. Of course, our ability to compute cross products endows (1) with computational value. The following results are evident.

THEOREM 3.1. *Let* A *be any nonsingular* $n \times n$ *matrix. Then*

$$_1(A^{-1}) = \frac{1}{|A|}\times(A_2, \cdots, A_n),\ _2(A^{-1}) = \frac{-1}{|A|}\times(A_1, A_3, \cdots, A_n), \cdots,\ _n(A^{-1})$$
$$= \frac{(-1)^{n-1}}{|A|}\times(A_1, \cdots, A_{n-1}).$$

THEOREM 3.2. *Let A be any nonsingular $n\times n$ matrix. Then*

$$(A^{-1})_1 = \frac{1}{|A|}\times({}_2A,\cdots,{}_nA),\ (A^{-1})_2 = \frac{-1}{|A|}\times({}_1A,{}_3A,\cdots,{}_nA),\cdots,$$

$$(A^{-1})_n = \frac{(-1)^{n-1}}{|A|}\times({}_1A,\cdots,{}_{n-1}A).$$

These results are valuable because they enable us to compute each column (or row) vector of the inverse of a nonsingular $n\times n$ matrix by row-reducing an $(n-1)\times n$ matrix. A standard method of computing inverses involves row-reducing one $n\times 2n$ matrix. From the viewpoint of an automatic computer the advantage of the present method is its economy of memory (think of n as large). The idea is to use a computer to calculate each cross product. Of course, $|A|$ is easily calculated since $|A| = A_1\cdot\times(A_2,\cdots,A_n)$, a dot product. We illustrate this technique taking n small, $n=3$.

Example 2. Calculate A^{-1} where

$$A = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \\ 0 & 2 & 5 \end{bmatrix}.$$

Solution. We compute ${}_1(A^{-1})$, ${}_2(A^{-1})$, and ${}_3(A^{-1})$ in turn. Now

$$\begin{pmatrix} -1 & 1 & 1 \\ 0 & 2 & 5 \end{pmatrix} \sim \begin{pmatrix} -1 & 0 & -1.5 \\ 0 & 2 & 5 \end{pmatrix};$$

so $\times((-1, 1, 1), (0, 2, 5)) = (3, 5, -2)$; thus ${}_1(A^{-1}) = -1(3, 5, -2) = (-3, -5, 2)$. Considering

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 2 & 5 \end{pmatrix}$$

we see that $\times((1, 0, 2), (0, 2, 5)) = (-4, -5, 2)$; thus ${}_2(A^{-1}) = (-4, -5, 2)$. Finally,

$$\begin{pmatrix} 1 & 0 & 2 \\ -1 & 1 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix};$$

so $\times((1, 0, 2), (-1, 1, 1)) = (-2, -3, 1)$; thus ${}_3(A^{-1}) = (2, 3, -1)$. We conclude that

$$A^{-1} = \begin{bmatrix} -3 & -4 & 2 \\ -5 & -5 & 3 \\ 2 & 2 & -1 \end{bmatrix}.$$

4. A second method of computing inverses. Let A be a nonsingular matrix and let B be a matrix obtained from A by applying a permitted row-reduction operation, i.e., B is obtained from A by one of the following operations:

(i) Interchange two row vectors.

(ii) Multiply a row vector by a nonzero scalar.

(iii) Add one row vector to another row vector.

We wish to pin down the relationship between A^{-1} and B^{-1}. Using Theorem 3.1 and the lemmas of Section 2 we easily verify the following statements.

LEMMA 4.1. *Let B be obtained from a nonsingular matrix A by interchanging A_r and A_s. Then the column vectors of A^{-1} can be obtained from the column vectors of B^{-1} by interchanging ${}_r(B^{-1})$ and ${}_s(B^{-1})$.*

LEMMA 4.2. *Let B be obtained from a nonsingular matrix A by replacing A_r by kA_r, where $k \neq 0$. Then*

$$
{}_i(A^{-1}) = \begin{cases} k\,{}_r(B^{-1}) & \text{if } i = r \\ {}_i(B^{-1}) & \text{otherwise.} \end{cases}
$$

LEMMA 4.3. *Let B be obtained from a nonsingular matrix A by replacing A_r by $A_r + A_s$, where $r \neq s$. Then*

$$
{}_i(A^{-1}) = \begin{cases} {}_s(B^{-1}) + {}_r(B^{-1}) & \text{if } i = s \\ {}_i(B^{-1}) & \text{otherwise.} \end{cases}
$$

LEMMA 4.4. *Let B be obtained from a nonsingular matrix A by replacing A_r by $A_r + kA_s$, where $r \neq s$. Then*

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{}_i(A^{-1}) = \begin{cases} {}_s(B^{-1}) + k\,{}_r(B^{-1}) & \text{if } i = s \\ {}_i(B^{-1}) & \text{otherwise.} \end{cases}
$$

The idea is to row-reduce a given matrix, say A, until we obtain a matrix, say B, whose inverse is known. By use of the preceding lemmas, the row operations that produce B generate a sequence of column operations that produce A^{-1} from B^{-1}. Now a square matrix of the form

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 12 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix}
$$

in which each row and column has exactly one nonzero entry, is readily invertible. Indeed, its inverse is obtained by replacing each nonzero entry by its reciprocal and transposing the resulting matrix. For example,

$$
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1/4 \\ 0 & 1/12 & 0 & 0 \end{bmatrix}
$$

is the inverse of the above matrix. This observation makes it possible to simplify our row-reduction procedure by using only Lemma 4.4. We choose a nonzero entry from ${}_1A$ and use it to transform the remaining entries of ${}_1A$ into zero; this requires $n-1$ row transformations of the type involved in Lemma 4.4. So, by this lemma, A^{-1} can be obtained by adding to the first column vector of the inverse of the resulting matrix, a linear combination of its other column vectors. We repeat the procedure on the remaining columns of the resulting matrix, using as our "pivot" an entry from a row not previously used in this capacity. So we obtain a matrix of the form displayed above, whose inverse we can readily compute. Systematically modifying the column vectors of its inverse (by adding a linear combination of its other column vectors) we obtain A^{-1}.

There are two items that we must record during the reduction process:

a. the pivotal entry for each column

b. the multipliers used to reduce the column.

We shall use an $n\times n$ matrix to record this information, circling the pivotal entries. Let us illustrate our technique.

Example 3. Compute

$$\begin{bmatrix} 0 & 2 & 5 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix}^{-1}.$$

Solution. Our first pivot is the entry in (2, 1); the second pivot is the entry in (3, 2) of the resulting matrix; the third pivot is the entry in (1, 3) of the resulting matrix. We now display the row-reduction steps:

$$A = \begin{bmatrix} 0 & 2 & 5 \\ 1 & 0 & 2 \\ -1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & 5 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = B.$$

The key data is recorded in the following matrix:

$$\begin{bmatrix} 0 & -2 & \textcircled{-1} \\ \textcircled{1} & 0 & 2 \\ 1 & \textcircled{1} & 3 \end{bmatrix}. \tag{2}$$

This means that:

a. The entry in (2, 1) of A is the pivot for the first column; the entry in (1, 1) transformed into 0 by multiplying the pivotal row by 0 and adding to the first row; the entry in (3, 1) is transformed into 0 by multiplying the pivotal row by 1 and adding to the third row.

b. The entry in (3,2) of the resulting matrix is the pivot for the second column; the entry in (1, 2) is transformed into 0 by multiplying the pivotal row by -2 and adding to the first row; the entry in (2, 2) is transformed into 0 by multiplying the pivotal row by 0 and adding to the second row.

c. The entry in (1, 3) of the resulting matrix is the pivot for the third column; the entry in (2, 3) is transformed into 0 by multiplying the pivotal row by 2 and adding to the second row; the entry in (3, 3) is transformed into 0 by multiplying the pivotal row by 3 and adding to the third row.

We shall use this information to get A^{-1} from B^{-1}. Observe that

$$B^{-1} = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix} = C.$$

Now, the entries in the last column of (2) direct us to replace ${}_1C$ by ${}_1C+2\,{}_2C+3\,{}_3C$. This yields

$$D = \begin{pmatrix} 2 & 1 & 0 \\ 3 & 0 & 1 \\ -1 & 0 & 0 \end{pmatrix}.$$

Considering the second column of (2) we replace ${}_3D$ by ${}_3D-2\,{}_1D$; we obtain

$$E = \begin{pmatrix} 2 & 1 & -4 \\ 3 & 0 & -5 \\ -1 & 0 & 2 \end{pmatrix}.$$

Finally, in view of the first column of (2), we replace ${}_2E$ by ${}_2E+{}_3E$; the resulting matrix is

$$\begin{pmatrix} 2 & -3 & -4 \\ 3 & -5 & -5 \\ -1 & 2 & 2 \end{pmatrix}.$$

By Lemma 4.4, this is the required inverse.

It is convenient to carry out the required column modifications in the manner of an automatic computer, actually erasing the column involved and replacing it by the modified column. Alternatively, this can be achieved by exhibiting a single, modified matrix directly under the initial inverse.

Example 4. Compute

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix}^{-1}.$$

Solution. First, we row-reduce the given matrix:

$$\begin{pmatrix} 1 & 0 & -1 & 0 \\ 2 & 0 & 0 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 0 & 3/2 \\ 0 & 0 & 2 & 3 \\ 0 & 1 & 0 & -3/2 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

The key data is recorded in the following matrix:

$$\begin{bmatrix} \textcircled{1} & 0 & 1/2 & -1/2 \\ -2 & 0 & \textcircled{2} & -1 \\ 0 & \textcircled{1} & -1/2 & 1/2 \\ 0 & -2 & 1 & \textcircled{3} \end{bmatrix}.$$

Accordingly, we modify the columns of the following matrix in the order 4, 2, 3, 1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \\ -2/3 & 1/3 & -2/3 & 1/3 \end{bmatrix}.$$

The preceding matrix is the required inverse.

Finally, we mention that the relationships contained in the lemmas of this section can be established by observing that a row-reduction operation can be achieved by premultiplying by a suitable elementary matrix. The fact is, however, that the algebraic properties of $\times$ strongly suggest the existence of the simple relationships contained in these lemmas; whereas to the extent that they lack a developed algebra, elementary matrices fail to be suggestive.

SOME GENERALIZATIONS OF WYTHOFF'S GAME AND OTHER RELATED GAMES

JOHN C. HOLLADAY, University of California, Irvine

Wythoff's game, described below, although not as well known as Nim, has nevertheless attracted a considerable degree of interest. It has probably been a major stimulus for the study of complementary sequences. When it was first presented by Wythoff [1], its analysis was given only inductively, that is, as complementary sequences with specified differences. It wasn't until later that its analysis was expressed in closed form [2].

A position in Wythoff's game consists of two nonnegative integers, which may be thought of as piles of counters. A move by either of the two players consists of either (a) reducing the size of but a single pile or (b) reducing both piles by the same amount. The player leaving no counters after his move wins the game.

It is well known that for games with this kind of symmetry (i.e., both players make the same type of moves) in which the terminal player wins, positions may be categorized as safe or unsafe. A perfect player is insured of a win if he is left an unsafe position or if he leaves his opponent a safe one. A dichotomous labeling of positions as safe or unsafe is the one that satisfies the above defining

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Accordingly, we modify the columns of the following matrix in the order 4, 2, 3, 1:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 1/3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 1 & -1/2 \\ 0 & 0 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \\ -2/3 & 1/3 & -2/3 & 1/3 \end{bmatrix}.$$

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It is well known that for games with this kind of symmetry (i.e., both players make the same type of moves) in which the terminal player wins, positions may be categorized as safe or unsafe. A perfect player is insured of a win if he is left an unsafe position or if he leaves his opponent a safe one. A dichotomous labeling of positions as safe or unsafe is the one that satisfies the above defining properties if and only if it satisfies the following two requirements:

(1) For each unsafe position, there exists a move that yields a safe position. position. Note that this requirement implies that a final position is safe.

(2) Any move from a safe position leads to an unsafe position.

(A proof of this is presented in [**3**].)

Let G_1, G_2, and G_3 be three (not necessarily different) such termination games. Let the games have the same set of possible positions. Let any admissible move in G_1 be admissible as a move in G_2. Similarly, let any move in G_2 be a move in G_3. Then in view of the above, it is easily seen that a labeling of positions as safe or unsafe is valid for all three games if and only if (1) is valid for G_1 and (2) is valid for G_3.

For each of the games of this paper, a position consists of two piles. A nonnegative integer k is assumed to be given and for each type of game, a value of k defines a specific game. In describing moves, the numbers x and y refer to the sizes of the piles before the move and x^* and y^* to their respective sizes after the move. In all of the games, the ordering of x and y is unimportant in that a transition from (x, y) to (x^*, y^*) is a valid move if and only if one from (y, x) to (y^*, x^*) is also a move. Hence, the set of all safe positions is fully characterized by specifying the set of those safe positions (x, y) satisfying the condition $x \leqq y$.

The following four games, A, B, C, and D all have the same set of safe positions. Game A has been singled out for attention because it seems to be aesthetically more appealing, due in large part to how simply it is defined. Under each game is listed the types of moves permitted for it. A type of move is defined by the set of requirements that such a move must satisfy. For each type of move, the requirements for such a move are stated verbally. Then they are repeated in terms of equations and inequalities that x^* and y^* must satisfy.

Game A:

Type 1. Take from either pile but not from both. $x^* = x$ and $y^* < y$ or $y^* = y$ and $x^* < x$.

Type 2. Take from either or both piles, but do not take more than k more counters from one pile than from the other. $x^* \leqq x$ and $y^* \leqq y$; $x^* + y^* < x + y$; $|(x - x^*) - (y - y^*)| \leqq k$.

Game B:

Type 1. See above.

Type 3. Take from one or both piles, but do not take more counters from the smaller pile than from the larger pile. Also, do not take more than k more counters from the larger pile than from the smaller pile, or enough more that a largest pile becomes the smaller pile. $x^* \leqq x$ and $y^* \leqq y$; $x^* + y^* < x + y$. If $x \leqq y$, then $x^* \leqq y^*$ and $y - y^* - k \leqq x - x^* \leqq y - y^*$. If $y \leqq x$, then $y^* \leqq x^*$ and $x - x^* - k \leqq y - y^* \leqq x - x^*$.

Game C:

Type 1. See above.

Type 4. Take from the piles and arrange the remaining counters so that the change in size of one pile does not differ from the change in size of the other pile by more than k counters. $x^*+y^*<x+y$; $|(x-x^*)-(y-y^*)|\leqq k$.

Type 5. Leave the same total number of counters, but make the position more lopsided. Equivalently, move one or more counters from a smallest pile to the other pile. $x^*+y^*=x+y$; $|x^*-y^*|>|x-y|$.

Game D:

Type 1 or *Type* 4. See above.

Type 6. Leave the same total number of counters, but make the position less lopsided. $x^*+y^*=x+y$; $|x^*-y^*|<|x-y|$.

It is asserted that the safe positions are the same for all four games. Also, it is easily seen that any move in B is a move in A and that any move in A is a move in either C or D. Hence, to prove this assertion and to demonstrate a characterization of which positions are safe, it is sufficient to show the following two requirements:

$(1)_B$ For each unsafe position, there exists a move in Game B that yields a safe position.

$(2)_{C,D}$ Any move of Game C or D from a safe position leads to an unsafe position.

For either A or B, Wythoff's game is obtained by setting k equal to zero. Note that the moves of C and D cannot be combined to form an even less restrictive termination game since the play of such a game need never terminate. However, a perfect player on being left an unsafe position, could insure himself of a victorious termination in the game combining the moves of C and D by restricting himself to winning moves of *Types* 1 and 4.

Let two sequences $a(\cdot)$ and $b(\cdot)$ be defined inductively as follows: For $n>0$, $a(n)$ is the smallest positive integer not contained in the set $\{i\,|\,i=\text{either } a(j) \text{ or } b(j) \text{ for some } 0<j<n\}$. Let $b(n)$ be defined by

$$b(n) = a(n) + (k+1)n. \tag{3}$$

In other words, $a(\cdot)$ and $b(\cdot)$ are those complementary, monotonic increasing sequences that satisfy (3). Also define $a(0)=b(0)=0$. Then the set of safe positions for games A, B, C or D are of the form $(a(n), b(n))$ and $(b(n), a(n))$ where n ranges over the set of nonnegative integers.

To show that (1) holds for Game B, let (x, y) be some unsafe position. Without loss of generality, let $x\leqq y$. Then for some n, either

$$a(n) \leqq b(n) = x \leqq y \neq a(n) \tag{4}$$

or

$$a(n) = x \quad \text{and} \quad y > b(n) \tag{5}$$

or

$$a(n) = x \leqq y < b(n) = x + (k+1)n. \tag{6}$$

In case of (4), a transition to the safe position $(x, a(n))$ is a move of *Type* 1.

In case of (5), a transition to the safe position $(x, b(n))$ is a move of *Type* 1. In case of (6), define $m<n$ as that integer such that

$$(k+1)m \leqq y - x < (k+1)(m+1). \tag{7}$$

Then a transition to the safe position $(a(m), b(m))$ is a move of *Type* 3.

To see that (2) is valid for games C and D, note first that because $a(\cdot)$ and $b(\cdot)$ were complementary, the following is true: If both (x, y) and (x, z) are safe positions, then $y=z$. Therefore, moves of *Type* 1 send safe positions into unsafe positions.

If m and n are nonnegative integers, then, from (3), if

$$\left| b(m) - a(m) - b(n) + a(n) \right| \leqq k, \tag{8}$$

then $m=n$. Therefore, moves of *Type* 4 send safe positions into unsafe positions.

Since both $a(\cdot)$ and $b(\cdot)$ are monotonic increasing sequences,

$$a(m) + b(m) = a(n) + b(n) \tag{9}$$

implies that $m=n$. Therefore, moves of *Type* 5 or 6 send safe positions into unsafe positions.

Explicit formulas for the sequences $a(\cdot)$ and $b(\cdot)$ for characterizing safe positions in A, B, C and D are given by

$$a(n) = [(1 - k + \sqrt{(k+1)^2 + 4})n/2] \tag{10}$$

and

$$b(n) = [(3 + k + \sqrt{(k+1)^2 + 4})n/2] \tag{11}$$

where for r a real number, $[r]$ is defined as that integer such that $[r] \leqq r < [r] + 1$. These formulas may be obtained from Beatty's result, [**4**], or from [**5**] (Example 1 of Section IV; set $\alpha = (k+1)/2$ and $\beta = 0$).

The remaining games to be discussed in this paper are as follows, with more attention being focused on Games E, F and G. Shortened verbal definitions of moves of *Types* 1, 5, and 6 are repeated for convenience.

Game E:

Type 1. Take from any one pile.

Type 7. Take from either or both piles, but do not take a total of more than k counters. $x^* \leqq x$ and $y^* \leqq y$; $0 < (x+y) - (x^*+y^*) \leqq k$.

Game F:

Type 1. See above.

Type 8. Take no more than k counters, and arrange the remaining counters. $0 < (x+y) - (x^*+y^*) \leqq k$.

Game G:

Type 1 or 8. See above.

Type 5. Make the position more lopsided.

Game H:

Type 6. Make the position less lopsided.

Type 8. See above.

Game I:

Type 1, 6, or 8. See above.

Game J:

Type 5 or 8. See above.

The two games, E and F, have the same set of safe positions. Namely, a position (x, y) is safe if and only if for some $n \geqq 0$, either

$$x = y = (k+1)n \tag{12}$$

or

$$x > (k+1)n, \quad y > (k+1)n \quad \text{and} \quad x + y = (k+1)(2n+1). \tag{13}$$

Note that when $k=0$ or 1, all moves are of *Type* 1 and the game degenerates to two-pile Nim.

To show that (1) is valid for the more restrictive Game E, let (x, y) be any unsafe position. Without loss of generality, let $x \leqq y$. Define n as that integer such that

$$(k+1)n \leqq x < (k+1)(n+1). \tag{14}$$

Then either

$$x = (k+1)n < y \tag{15}$$

or

$$x > (k+1)n \quad \text{and} \quad x + y < (k+1)(2n+1) \tag{16}$$

or

$$x > (k+1)n \quad \text{and} \quad x + y > (k+1)(2n+1). \tag{17}$$

In case of (15), a move of *Type* 1 may be made to the safe position $((k+1)n, (k+1)n)$. In case of (16), a move of *Type* 7 may be made to this same safe position. In case of (17), a move of *Type* 1 may be made to the safe position $(x, (k+1)(2n+1)-x)$.

To see that (2) is valid for the less restrictive Game F, note that for each x there is only one y such that (x, y) is safe. Hence any move of *Type* 1 changes a safe position to an unsafe position. Also, whenever (x, y) is safe, $x+y \equiv 0 \pmod{k+1}$. Therefore, any move of *Type* 8 changes safe positions into unsafe positions.

It is easily seen that when $k \leqq 2$, no move of *Type* 5 may change any safe position for Game F into another one. Therefore, when $k \leqq 2$, the safe positions for Game G are the same as for Game E or F. However, when $k > 2$, a transition from the safe (for Game F) position $((k+1)n+2, (k+1)n+k-1)$ to the safe position $((k+1)n+1, (k+1)n+k)$ is a move of *Type* 5. So when $k > 2$, the set of safe positions for Game G must be different from the set for Game E or F.

Whenever $k \geqq 3$, there exists a unique pair of complementary increasing sequences $p(\cdot)$ and $q(\cdot)$ such that for each $n > 0$,

$$p(n) < q(n) = (k+1)n - p(n). \tag{18}$$

Furthermore, the values of p and q are given by the following expressions:

$$p(n) = [((k+1)n + 1 - \sqrt{(k+1)(kn - 3n + 2)n})/2]. \tag{19}$$

$$q(n) = [((k+1)n + 1 + \sqrt{(k+1)(kn - 3n + 2)n})/2]. \tag{20}$$

These results are obtained from [5] (Example 3 of Section IV; set $\alpha = (k-1)/2$ and $\beta = 1/2$). Extend the definition of $p(\cdot)$ and $q(\cdot)$ so that $p(0) = q(0) = 0$. Then it will be shown that the safe positions for Game G are the positions of the form $(p(n), q(n))$ or the form $(q(n), p(n))$.

To show that (1) holds for G, let (x, y) be any unsafe position. Without loss of generality, let $x \leqq y$. Then for some $n \geqq 0$, either

$$x = p(n), \quad y > q(n) \tag{21}$$

or

$$x = p(n), \quad y < q(n) \tag{22}$$

or

$$x = q(n). \tag{23}$$

In case of (21), the safe position $(x, q(n))$ is obtainable by a move of *Type* 1. In case of (22), the safe position (x^*, y^*) is obtainable where

$$x^* = p([(x+y)/(k+1)]) \quad \text{and} \quad y^* = q([(x+y)/(k+1)]). \tag{24}$$

If $x + y \equiv 0 \pmod{k+1}$, the move to (x^*, y^*) is of *Type* 5, but if $x + y \not\equiv 0 \pmod{k+1}$, the move is of *Type* 8. In case of (23), the safe position $(q(n), p(n))$ is obtainable by a move of *Type* 1.

To show that (2) holds for G, let (x, y) be any safe position. Since $p(\cdot)$ and $q(\cdot)$ were complementary sequences, no move of *Type* 1 applied to (x, y) may yield another safe position. Item (18) implies that the only other safe position having the same pile sum $x + y$ is (y, x). Hence, no move of *Type* 5 may change (x, y) to another safe position. Also, (18) implies that $x + y \equiv 0 \pmod{k+1}$. So no move of *Type* 8 may yield another safe position.

When $k \geqq 1$, a position (x, y) is safe in either Game H or Game I if and only if for some $n \geqq 0$,

$$\min(x, y) = [(k+1)n/2] \quad \text{and} \quad \max(x, y) = (k+1)n - \min(x, y). \tag{25}$$

For Game J, a position (x, y) is safe if and only if for some $n \geqq 0$,

$$\min(x, y) = 0 \quad \text{and} \quad \max(x, y) = (k+1)n. \tag{26}$$

The proofs of the statements of this paragraph, which are straightforward applications of (1) and (2), are left as exercises for the reader. For Games H and J, when $k=0$, let any player faced with a position for which there is no move lose. Then even when $k=0$, (25) and (26) characterize the safe positions for Games H and J respectively. If $k=0$, a position (x, y) is safe in Game I if and only if $x=y$.

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RECURRENT SEQUENCES AND PASCAL'S TRIANGLE

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A. Introduction. The Fibonacci sequence can be found by summing the terms on successive "diagonals" of Pascal's Triangle [1]. J. Raab [2] generalized this procedure to show other sets of parallel diagonals generating different recurrent sequences. This generalization is essentially the same as Phase One in what is to follow. The purpose of this paper is to show that there exist infinitely many more recurrent sequences within Pascal's Triangle by summing the terms on diagonals of different slopes. Each sequence shall be of the type such that each term is the sum of *two* former terms. There is also a unique relationship between just what two terms are involved and the slope of the diagonals being considered.

For this purpose it is convenient to arrange the terms of Pascal's Triangle on the point-lattice determined by the nonnegative integral points of a rectangular coordinate system. (See Figure 1.)

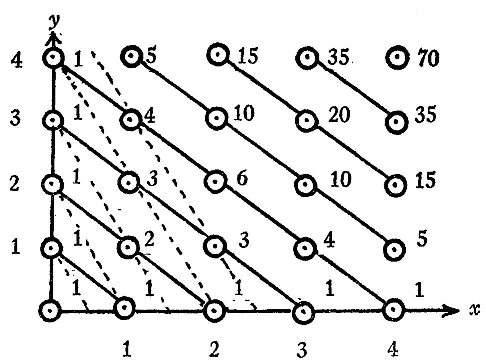

FIG. 1

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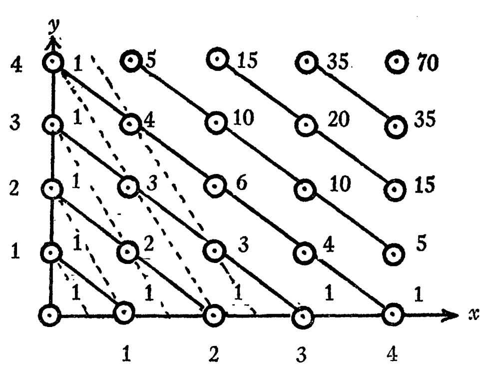

FIG. 1

With this arrangement the coordinates, (x, y), of the lattice point uniquely determine the location and value of a Pascal number $\binom{n}{r}$. The value is seen to be

A(1) $$\binom{n}{r} = \frac{(x+y)!}{x!\cdot y!},$$

since $n=x+y$ and $r=y$ (or x, because of the symmetry involved).

B. Phase one. Consider the linear equation

B(1) $$x + y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

This equation represents the nth row (diagonal) of Pascal's Triangle. If we sum the Pascal numbers on each row determined by B(1) for successive values of n, we obtain the sequence

B(1.1) $$1, 2, 4, 8, \cdots, 2^n, \cdots$$

whose recurrence relation is given by

B(1.2) $$P_n = P_{n-1} + P_{n-1},$$

where $P_0, P_1, \cdots, P_n, \cdots$ denote the terms of the sequence, and the formula for the nth term is given by

B(1.3) $$P_n = 2^n = \sum_{\substack{x=0,y=0\\ x+y=n}}^{n,n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{n} \binom{n}{r}.$$

(*Note*: the nth term is the term formed by summing all of the Pascal numbers on the line $x+y=n$ and, if we were counting the terms, this term would actually be the $(n+1)$th term in the sequence.)

The sum of the first n terms of the sequence is given by

B(1.4) $$\sum_{k=0}^{n-1} P_k = P_n - 1.$$

Now consider the linear equation

B(2) $$2x + y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

This equation represents the nth diagonal referred to above used to obtain the nth Fibonacci number. By summing the Pascal numbers on each diagonal determined by B(2) for successive values of n (see the dotted lines, Figure 1), we obtain the sequence

B(2.1) $$1, 1, 2, 3, 5, \cdots, F_n, \cdots$$

whose recurrence relation is given by

B(2.2) $$F_n = F_{n-2} + F_{n-1}.$$

The formula for the nth term is given by

B(2.3)
$$F_n = \sum_{\substack{x=0,y=0 \\ 2x+y=n}}^{[n/2],n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{[n/2]} \binom{n-r}{r}$$

where [ ] denotes the greatest integer function and the sum of the first n terms of the sequence is given by

B(2.4)
$$\sum_{k=0}^{n-1} F_k = F_{n+1} - 1.$$

Next consider the linear equation

B(3)
$$3x + y = n \qquad \text{for } n = 0, 1, 2, \cdots.$$

In a way similar to that used above we establish the sequence

B(3.1)
$$1, 1, 1, 2, 3, 4, 6, 9, \cdots, G_n, \cdots$$

whose recurrence relation is given by

B(3.2)
$$G_n = G_{n-3} + G_{n-1}.$$

The formula for the nth term is given by

B(3.3)
$$G_n = \sum_{\substack{x=0,y=0 \\ 3x+y=n}}^{[n/3],n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{[n/3]} \binom{n-2r}{r}$$

and the sum of the first n terms of the sequence is given by

B(3.4)
$$\sum_{k=0}^{n-1} G_k = G_{n+2} - 1.$$

Now consider the linear equation

B(j)
$$jx + y = n \qquad \text{for } n = 0, 1, 2, \cdots \quad \text{and } j = 1, 2, 3, \cdots.$$

This equation, by the procedure referred to above, establishes a sequence whose recurrence relation is given by

B(j.2)
$$T_n = T_{n-j} + T_{n-1}.$$

The formula for the nth term is given by

B(j.3)
$$T_n = \sum_{\substack{x=0,y=0 \\ jx+y=n}}^{[n/j],n} \frac{(x+y)!}{x!\cdot y!} = \sum_{r=0}^{[n/j]} \binom{n-(j-1)r}{r}$$

and the sum of the first n terms of the sequence is given by

B(j.4)
$$\sum_{k=0}^{n-1} T_k = T_{n+(j-1)} - 1.$$

For a proof of B(j.2) and B(j.4), see E(a.2) and E(a.4).

C. Phase two. In Phase One each linear equation had the coefficient pair $\langle j, 1\rangle$, giving rise to infinitely many recurrent sequences. We now consider any coefficient pair $\langle j, 2\rangle$, determining the following equations:

C(1) $x + 2y = n$

C(2) $2x + 2y = n$

C(3) $3x + 2y = n$ for $n = 0, 1, 2, \cdots$.

$\vdots$

C(j) $jx + 2y = n$

Several of these are equivalent to cases already discussed, namely,

(i) any coefficient pair $\langle a, b\rangle$ will yield the same sequence and recurrence relation as the pair $\langle b, a\rangle$, because of the symmetry of the Pascal Triangle about the line $y=x$;

(ii) any coefficient pair $\langle a, b\rangle$ presupposes the fact that a and b are relatively prime, since if they are not, then two possibilities occur. Either the equation will be reduced by dividing thru by the greatest common divisor or it will be such that the given value of n will not be divisible by the g.c.d. of a and b. If the equation is reduced, it will have been treated in an earlier phase, and if the equation cannot be reduced, there will be no integral solutions (see [**3**]), and thus no sequence will be determined;

(iii) even when a and b are relatively prime, there will be cases where $ax+by=n$ will not have nonnegative integral solutions. This means that for those particular values of n, the recurrent sequence derived from $ax+by=n$ will have zero as a value for those nth terms in the sequence, since there will be no Pascal numbers to sum. Thus we establish the following useful

LEMMA. *The equation $ax+by=n$, where a, b, and n are nonnegative integers and $ab\neq 0$ and $(a, b)=1$, will not have nonnegative integral solutions when $n=ab-(ja+kb)$, where $j, k=1, 2, 3, \cdots$.*

Proof. Assume $n=ab-(ja+kb)$ so that $ax+by=ab-ja-kb$ with nonnegative solution (x, y). Thus $a(x+j)+b(y+k)=ab$. Let $X=x+j$ and $Y=y+k$; then $aX+bY=ab$. It is important to note here that both X and Y, as well as both a and b, are greater than or equal to one. We can now transform the above equation to $b=X+(bY)/a$ or $b-X=(bY)/a$. Now $b-X$ is an integer; therefore a divides Y since a and b are relatively prime. Suppose $Y/a=r$ so that $Y=ar$.

Similarly we can show that $a-Y=(aX)/b$ and hence conclude that b divides X. Suppose $X/b=s$ so that $X=bs$. Then, by substitution, we have $abs+abr=ab$ or $ab(r+s)=ab$. Therefore $r+s=1$. But both r and s are greater than or equal to one; therefore we have a contradiction.

Thus, in view of the above discussion, the first new case in Phase Two is C(3):

C(3) $3x + 2y = n$ for $n = 0, 1, 2, \cdots$.

By summing the Pascal numbers on each diagonal determined by C(3) for successive values of n we find that there is no positive integral solution for $n=1$, since, $1=3\cdot 2-(3+2)$ as predicted by the lemma; therefore the sequence is

C(3.1) $$1, 0, 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, \cdots, H_n, \cdots$$

whose recurrence relation is given by

C(3.2) $$H_n = H_{n-3} + H_{n-2}.$$

The formula for the nth term is given by

C(3.3) $$H_n = \sum_{\substack{x=0,y=0 \\ 3x+2y=n}}^{[n/3],[n/2]} \frac{(x+y)!}{x!\cdot y!}.$$

A formula for H_n in terms of n and r could also be given; however, it actually requires *two* formulas and, in general, the formula will require b different representations, one for each of the different values in the residue class of n (mod b). Phase Three will need three formulas, etc. More will be said about this in the discussion of the general phase.

The formula for the sum of the first n terms of the sequence is given by

C(3.4) $$\sum_{k=0}^{n-1} H_k = H_{n+2} + H_{n+1} - 1.$$

Now by similar considerations of the next new case,

C(5) $$5x + 2y = n \qquad \text{for } n = 0, 1, 2, \cdots,$$

we find that there are no solutions for $n=1$ and $n=3$, since $1=5\cdot 2-(5+2\cdot 2)$ and $3=5\cdot 2-(5+2)$. Hence, by summing the Pascal numbers on each successive diagonal determined by C(5), we obtain the sequence

C(5.1) $$1, 0, 1, 0, 1, 1, 1, 2, 1, 3, 2, 4, 4, 5, 7, 7, 11, 11, \cdots, I_n, \cdots$$

whose recurrence relation is given by

C(5.2) $$I_n = I_{n-5} + I_{n-2}.$$

The formula for the nth term is given by

C(5.3) $$I_n = \sum_{\substack{x=0,y=0 \\ 5x+2y=n}}^{[n/5],[n/2]} \frac{(x+y)!}{x!\cdot y!},$$

and the sum of the first n terms is given by

C(5.4) $$\sum_{k=0}^{n-1} I_k = I_{n+4} + I_{n+3} - 1.$$

Now consider the general case of Phase Two,

C(j) $$jx + 2y = n \qquad \text{for } n = 0, 1, 2, \cdots \qquad \text{and } j = 1, 2, 3, \cdots.$$

This equation establishes a sequence whose recurrence relation is given by

C(j.2) $$T_n = T_{n-j} + T_{n-2}$$

The formula for the nth term is given by

C(j.3) $$T_n = \sum_{\substack{x=0,y=0\\ jx+2y=n}}^{[n/j],[n/2]} \frac{(x+y)!}{x!\cdot y!}$$

and the sum of the first n terms of the sequence is given by

C(j.4) $$\sum_{k=0}^{n-1} T_k = T_{n+(j-1)} + T_{n+(j-2)} - 1.$$

D. Phase three. Consider the equation

D(j) $jx + 3y = n$ for $n = 0, 1, 2, \cdots$ and $j = 1, 2, 3, \cdots$

where the pair $\langle j, 3\rangle$ complies with the remarks made in section C. By summing the Pascal numbers on each diagonal given by D(j) for successive values of n we obtain the sequence whose recurrence relation is given by

D(j.2) $$T_n = T_{n-j} + T_{n-3}.$$

The formula for the nth term is given by

D(j.3) $$T_n = \sum_{\substack{x=0,y=0\\ jx+3y=n}}^{[n/j],[n/3]} \frac{(x+y)!}{x!\cdot y!},$$

and the sum of the first n terms is given by

D(j.4) $$\sum_{k=0}^{n-1} T_k = T_{n+(j-1)} + T_{n+(j-2)} + T_{n+(j-3)} - 1.$$

Proof of the formulas of Phase Two and Phase Three will be covered by the proofs in the general phase that follows.

E. Phase b. In general, the equation

E(a) $ax + by = n$ for $n = 0, 1, 2, \cdots$ and $a, b = 1, 2, 3, \cdots,$

where the pair $\langle a, b\rangle$ complies with the remarks made in section C, will, by summing the Pascal numbers on each diagonal for successive values of n, yield a recurrent sequence whose recurrence relation is given by

E(a.2) $$T_n = T_{n-a} + T_{n-b}.$$

Proof. The first term in the series representing T_n, as defined by E(a.3) below, will be $(x+y)!/(x!\cdot y!)$, where x and y satisfies E(a). In the notation of A(1) this will equal

$$\binom{x+y}{x}.$$

Suppose that this first solution of x and y is the one where x is minimum (and hence y is maximum); then the next solution would be $(x_{\min}+b, y_{\max}-a)$ and the next would be $(x_{\min}+2b,\ y_{\max}-2a)$, etc., until $y_{\max}-ra$ becomes $y_{\min}$, where r is the greatest integer in the quotient $n/(ab)$. (This is a modified form of a standard result of number theory; see, for example, [3].) Now if we let $k=x_{\min}+y_{\max}$, we can write the first few terms of T_n as follows

$$(1)\qquad T_n=\binom{k}{x}+\binom{k+b-a}{x+b}+\binom{k+2b-2a}{x+2b}+\cdots,$$

where the x refers to only $x_{\min}$.

Next we look at T_{n-a}. The first term of this series will be of the form $\binom{k'}{x'}$ where k' and x' are related to k and x above in the following manner. First we note that

$$(2)\qquad ax'+by'=n-a,\quad\text{and}\quad x'+y'=k'.$$

Now since a, b and n have all been fixed we find that $x'=x_{\min}-1$ and $y'=y_{\max}$ is a solution, which upon substitution satisfies $ax+by=n$. Furthermore, since y' is the same $y_{\max}$ as found in the consideration of T_n, it will also be the maximum y in the consideration of T_{n-a}, since $n-a$ is less than n; hence x' is the corresponding minimum value. This makes $k'=k-1$. Therefore, we can write the first few terms of T_{n-a} as follows:

$$(3)\qquad T_{n-a}=\binom{k-1}{x-1}+\binom{k-1+b-a}{x-1+b}+\binom{k-1+2b-2a}{x-1+2b}+\cdots,$$

where, again, x refers to the original $x_{\min}$. If $x_{\min}$ is zero to begin with, then for the solution of (2) we choose $x'=x_{\min}+b-1$ and $y'=y_{\max}-a$ and this choice modifies (3) only to the extent that the first term is omitted in the series for T_{n-a}.

Next we consider T_{n-b}. The first term of this series will be of a form $\binom{k''}{x''}$ where

$$(4)\qquad ax''+by''=n-b,\quad\text{and}\quad x''+y''=k''.$$

We note here that

$$\binom{k''}{x''}=\binom{k''}{y''}$$

and we find that $x''=x_{\min}$ and $y''=y_{\max}-1$ is the appropriate solution of (4). Therefore, $k''=k-1$ and we can write the first few terms of T_{n-b} as follows:

$$(5)\qquad T_{n-b}=\binom{k-1}{x}+\binom{k-1+b-a}{x+b}+\binom{k-1+2b-2a}{x+2b}+\cdots,$$

where again x refers to the original $x_{\min}$. Now if we add the two series (3) and (5) termwise, we observe the result that we desired, namely, series (1), through the use of Pascal's Rule, which determines the very nature of Pascal's Triangle.

The general nth term of this sequence in terms of Pascal numbers is given by

E(a.3)
$$T_n = \sum_{\substack{x=0,y=0\\ ax+by=n}}^{[n/a],[n/b]} \frac{(x+y)!}{x!\cdot y!};$$

and the sum of the first n terms of this sequence is given by

E(a.4)
$$\sum_{k=0}^{n-1} T_k = T_{n+(a-1)} + T_{n+(a-2)} + \cdots + T_{n+(a-b)} - 1.$$

Proof (by induction). In the development of the phases above a was always greater than or equal to b. The remarks in section C indicate that this is an arbitrary choice because of the symmetry involved; however, one or the other choices must be made, but not both. We will assume here that $a \geqq b$. We divide the proof into two parts.

PART 1. We establish the formula for $n=1$. E(a.4) becomes

$$T_0 = T_a + T_{a-1} + T_{a-2} + \cdots + T_{a-b+1} - 1.$$

Now $T_0=1$, since $ax+by=0$ has only the single solution (0, 0) and $(0+0)!/(0!\cdot 0!)=1$. Also we see that $T_a=1$, since the only solution of $ax+by=a$ is (1, 0) and $(1+0)!/(1!\cdot 0!)=1$. In considering the other terms, $T_{a-1}, T_{a-2}, \cdots, T_{a-b+1}$, we find that the only solutions to the corresponding equations are, in all cases, $x=0$ and y equal to $(a-1)/b$, $(a-2)/b, \cdots, (a-b+1)/b$ respectively. Now only one number of this set of values is integral since the set $a, a-1, a-2, \cdots, a-b+1$ forms a residue class modulo b and, since a and b are relatively prime, the value a is omitted from consideration. All other solutions are nonintegral and therefore discarded and T_k for those values equals zero. Let k' be the one value that yields the integral solution and let k'' be that solution. Then

$$T_k = \frac{(0+k'')!}{0!\cdot k''!} = 1$$

Hence $T_0 = T_a + T_{k'} - 1$ or $1 = 1+1-1$ an identity.

PART 2. We assume the formula is true for n and show that then it is also true for $n+1$. To both sides of E(a.4) add T_n. Thus

$$\sum_{k=0}^{n} T_k = T_{n+(a-1)} + T_{n+(a-2)} + \cdots + T_{n+(a-b)} - 1 + T_n.$$

But from E(a.4) we have

$$\sum_{k=0}^{n} T_k = T_{n+1+(a-1)} + T_{n+1+(a-2)} + \cdots + T_{n+1+(a-b)} - 1.$$

We must show that the right members of the above equations are equal. Upon equating these two members and simplifying we have $T_{n+a-b} + T_n = T_{n+a}$. But we know from E(a.2) that $T_k = T_{k-a} + T_{k-b}$. Thus if $k = n+a$, we have the exact statement above.

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AN EXPLICIT EXPRESSION FOR BINARY DIGITAL SUMS

J. R. TROLLOPE, University of Alberta

1. Introduction and statement of results. If $\alpha(\kappa, r)$ denotes the sum of the digits of κ when κ is represented in base r, then it has been established ([1], [2], [3]) that

$$A(n, r) \equiv \sum_{\kappa<n} \alpha(\kappa, r) = \tfrac{1}{2}(r-1)n \log_r n - E(n, r) \tag{1.1}$$

where $E(n, r)=O(n)$. The purpose of this paper is to present a more detailed examination of $E(n, r)$ for the special case in which r is two. An interesting consequence of the investigation is that $E(n, 2)$ is expressed in terms of a continuous nondifferentiable function similar to that given by van der Waerden [4]. The function may be defined as follows: Let $g(x)$ be periodic of period one and defined on $[0, 1]$ by

$$g(x) = \begin{cases} \tfrac{1}{2}x, & 0 \leqq x \leqq \tfrac{1}{2} \\ \tfrac{1}{2}(1-x), & \tfrac{1}{2} < x \leqq 1. \end{cases} \tag{1.2}$$

The function

$$f(x) \equiv \sum_{i=0}^{\infty} \frac{1}{2^i} g(2^i x) \tag{1.3}$$

can be shown to be nondifferentiable. The relation between this function and $E(n, 2)$ is demonstrated in the following theorems:

THEOREM 1. *If the integer n is written $n=2^m(1+x)$, $0\leqq x<1$, then*

$$E(n, 2) = 2^{m-1}\{2f(x) + (1+x)\log_2(1+x) - 2x\}.$$

THEOREM 2. *If n is represented as in Theorem 1, then*

$$E(n, 2) < 2^{m-1}\{5/3 \log_2(5/3) - 2/3\}$$

and the constant cannot be reduced.

Drazin and Griffith [2] and more recently Clements and Lindstrom [5] have shown that

$$E(n, 2) < 2^{m-1}(1+x)\log_2(4/3) \tag{1.2}$$

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Drazin and Griffith [2] and more recently Clements and Lindstrom [**5**] have shown that

$$E(n, 2) < 2^{m-1}(1+x)\log_2(4/3) \tag{1.2}$$

where the constant $\log_2(4/3)$ is best possible relative to $2^{m-1}(1+x)$. However, if

$$(1.3)\qquad 1 + x > \left\{\frac{5/3 \log_2 5/3 - 2/3}{\log_2 (4/3)}\right\} = 1.348 \cdots,$$

then Theorem 2 produces a better result.

2. *Proof of Theorem* 1. For any positive integer κ, $\alpha(\kappa)$ denotes the digital sum of κ when it is represented in base two. The sum function $A(n)$ is defined by

$$(2.1)\qquad A(n) = \sum_{\kappa<n} \alpha(\kappa).$$

It has been shown [6], [7] that an explicit formula for $A(n)$ is

$$(2.2)\qquad A(n) = 1/2\left[mn + n_{m-1} + \sum_{i=0}^{m-2} (2a_{i+1} - 1)n_i\right]$$

where m is related to n by the binary expansion

$$(2.3)\qquad n = 2^m + a_{m-1}2^{m-1} + \cdots + a_1 2 + a_0,$$

the a_i being zero or one. The symbol n_i, $0 \leqq i < m$, is defined by

$$(2.4)\qquad n_i = a_i 2^i + a_{i-1}2^{i-1} + \cdots + 2a_1 + a_0.$$

We now show that the quantity

$$(2.5)\qquad R(n) \equiv 1/2\left[n_{m-1} - \sum_{i=0}^{m-2} (2a_{i+1} - 1)n_i\right]$$

is related to the van der Waerden function. First of all, it is easy to show that

$$(2.6)\qquad R(2n) = 2R(n).$$

This relation allows us to introduce a "normalized" function which is defined as follows: Let $x = p/2^m$ where p and m are nonnegative integers such that $0 \leqq p < 2^m$. Define the function $\phi(x)$ by

$$(2.7)\qquad \phi(x) = \frac{R(2^m + p)}{2^m}.$$

Note that $\phi(x)$ is uniquely defined, for if x has any other representation say $p'/2^{m'}$ then $p' = 2^{m'-m}p$. There is no loss of generality in assuming that $m' > m$. Hence

$$(2.8)\qquad 2^{m'} + p' = 2^{m'-m}(2^m + p)$$

where $m' - m$ is a positive integer. An iteration of equation (2.6) leads to

$$(2.9)\qquad \frac{R(2^{m'} + p')}{2^{m'}} = \frac{R(2^m + p)}{2^m}.$$

The function $\phi(x)$ is defined only on a subset of $[0, 1]$ and we now consider the problem of finding a continuous extension which is defined on the entire interval.

One approach to the problem is to consider the limit of a sequence of "polygonal" functions which identify with $\phi(x)$ on rationals of the form $p/2^m$. A fundamental building block in this sequence is a periodic function which we denote by $g(x)$. It is defined on the entire real line by

$$\text{(2.10)}\qquad \text{(i)}\quad g(x) = \begin{cases} \frac{1}{2}x, & 0 \leqq x \leqq \frac{1}{2}, \\ \frac{1}{2}(1-x), & \frac{1}{2} < x \leqq 1, \end{cases}$$

$$\text{(ii)}\quad g(x \pm 1) = g(x).$$

The sequence $\{f_m(x)\}$ is defined in the following rather awkward manner:

(2.11) (i) $f_m(x)$ is defined on $[0, 1]$ to be the function whose graph is the polygon joining the points $\{(0, 0), (\frac{1}{2^m}, \phi(\frac{1}{2^m})), \cdots, (p/2^m, \phi(p/2^m)), \cdots, (1, 0)\}$,

(ii) $f_m(x \pm 1) = f_m(x)$.

It can be shown that

$$\text{(2.12)}\qquad f_{m+1}(x) - f_m(x) = \frac{1}{2^m} g(2^m x).$$

The proof of this relation is straightforward and depends on the identity

$$\text{(2.13)}\qquad R(2^{m+1} + 2p + 1) - R(2^m + p + 1) - R(2^m + p) = 1/2,$$

a result which follows from (2.6) and the more elementary relation

$$\text{(2.14)}\qquad R(n+1) - R(n) = \tfrac{1}{2}(m+2) - \alpha(n).$$

Repeated iterations of (2.12) yield

$$\text{(2.15)}\qquad f_{m+1}(x) = \sum_{i=0}^{m} \frac{1}{2^i} g(2^i x).$$

Since $g(x)$ is bounded, the sequence $\{f_m(x)\}$ converges uniformly for all x. Hence the limiting function

$$\text{(2.16)}\qquad f(x) \equiv \sum_{i=0}^{\infty} \frac{1}{2^i} g(2^i x),$$

is a continuous extension of $\phi(x)$.

The van der Waerden function described by Titchmarsh [7] is defined relative to the base ten. However, a slight modification of the arguments used there shows that $f'(x)$ does not exist.

If we put $n = 2^m + p$, $0 \leqq p < 2^m$, then it follows from (2.2), (2.5), and (2.7) that

$$\text{(2.17)}\qquad A(n) = \tfrac{1}{2}mn + p - 2^m f(p/2^m),$$

and since

$$\text{(2.18)}\qquad \log_2 n = m + \log_2(1 + p/2^m)$$

we may write

$$A(n) = \tfrac{1}{2}n \log_2 n - 2^{m-1}\{2f(x) + (1 + x) \log_2 (1 + x) - 2x\} \tag{2.19}$$

where $x = p/2^m$. This completes the proof of Theorem 1.

3. *Proof of Theorem* 2. From (2.10) and (2.16) one can easily establish the following relations:

$$f(1/3) = f(2/3) = 1/3, \tag{3.1}$$

$$f(1/2 + \epsilon) = f(1/2 - \epsilon), \tag{3.2}$$

$$f(1/2 + \epsilon) \leqq 1/4 + 1/4f(4\epsilon), \tag{3.3}$$

where it is assumed that $0 \leqq \epsilon \leqq 1/2$. These relations imply that

$$\max f(x) = \max_{0 \leqq \epsilon \leqq 1/2} f(\epsilon) = 1/3. \tag{3.4}$$

The function $h(x)$ defined by

$$h(x) = (1 + x) \log_2 (1 + x) - 2x \tag{3.5}$$

is negative and concave upward over $[0, 1]$. Furthermore, one can use power series expansions to show that

$$h(1 - \epsilon) \geqq h(\epsilon), \qquad 0 \leqq \epsilon \leqq 1/3. \tag{3.6}$$

If we now put $F(x) = 2f(x) + h(x)$, then it is clear that

$$F(\epsilon) \leqq F(1 - \epsilon), \tag{3.7}$$

$$F(1/3 + \epsilon) \leqq \max \{F(1/3), F(2/3)\} = F(2/3), \tag{3.8}$$

provided $0 \leqq \epsilon \leqq 1/3$. Hence

$$M \equiv \max_{0 \leqq x \leqq 1} F(x) = \max_{0 \leqq \epsilon \leqq 1/3} F(2/3 + \epsilon). \tag{3.9}$$

To prove that $M = F(2/3)$ we consider the following special cases:

Case (*i*): $0 \leqq \epsilon \leqq 1/12$. The definition of $f(x)$ yields

$$f(2/3 + \epsilon) = 7/24 - \epsilon/2 + 1/8f(1/3 + 8\epsilon). \tag{3.10}$$

Hence, by (3.4), we have

$$f(2/3 + \epsilon) \leqq 1/3 - \epsilon/2. \tag{3.11}$$

Also

$$(5/3 + \epsilon) \log_2 (5/3 + \epsilon) \leqq 5/3 \log_2 5/3 + \epsilon\left(1 + \frac{1}{\log_2}\right), \tag{3.12}$$

and combining this result with (3.11) we obtain

$$F(2/3 + \epsilon) \leqq F(2/3) - \epsilon\left(2 - \frac{1}{\log_2}\right) \leqq F(2/3). \tag{3.13}$$

Case (*ii*): $1/12 \leqq \epsilon \leqq 1/3$. It is evident that

$$F(2/3+\epsilon) \leqq 2f(2/3+\epsilon) = 2f(3/4+\delta) \tag{3.14}$$

where $\delta = \epsilon - 1/12$. But

$$f(3/4+\delta) = \begin{cases} \dfrac{1}{4} + \dfrac{1}{16} f(16\delta), & 0 \leqq \delta \leqq 1/16, \\ \dfrac{1}{4} - \delta + \dfrac{1}{4} f(4\delta), & 1/16 \leqq \delta \leqq 1/4. \end{cases} \tag{3.15}$$

In either case

$$2f(3/4+\delta) \leqq \frac{13}{24} = .54166 \cdots \tag{3.16}$$

Since

$$F(2/3) > .56 \tag{3.17}$$

the proof of Theorem 2 is complete.

The above theorems which have been demonstrated for the base two can obviously be generalized to the base r and the results involve the corresponding van der Waerden-type functions. In view of the fact that the arguments are the same but the algebra is much more complicated, a discussion of the generalizations will be omitted.

Acknowledgements. I would like to express my appreciation to L. Moser for suggesting this problem and to H. Hansen for supplying numerical data on $E(n, 2)$.

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ON A PROPERTY OF LOGARITHMIC SPIRALS

N. X. VINH and A. ZIRAKZADEH, University of Colorado

Consider a point O on a right circular cylinder R. Place one tip of a compass on O and let the other tip, M, draw a curve on the surface of the cylinder. If b, the distance between the tips of the compass, is less than the diameter of the base of the cylinder, we will obtain a simple closed curve lying on the surface of the cylinder. Now if we had wrapped a sheet of paper around the cylinder, by opening this sheet of paper and placing it on a plane, we would obtain a simple closed plane curve. Furthermore, if b were reasonably small the curve would look somewhat like an ellipse.

Of course we can choose any curve C other than a circle as the base of the cylinder and obtain other simple closed plane curves. The purpose of this note is to find out under what conditions, if any, the curve obtained will be an ellipse. One necessary condition is that the cylinder, and consequently C, be symmetric with respect to a plane passing through the generator of the cylinder at O.

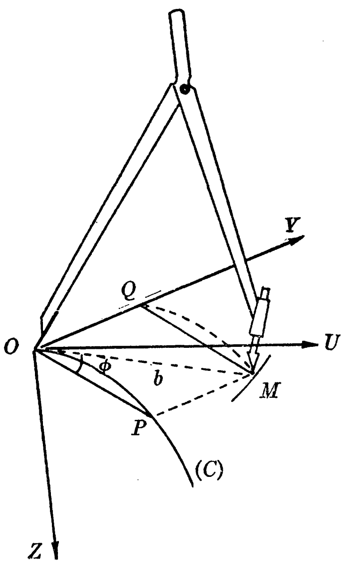

FIG. 1

Consider a right cylinder whose base is some plane curve C. Introduce the following coordinate system (Figure 1): Let OY be the generator of the cylinder passing through O, OU perpendicular to the above-mentioned plane of symmetry, and OZ perpendicular to OY and OU. The positive directions on each axis are chosen arbitrarily. The intersection of the cylinder with plane UOZ is, of course, the curve C, the base of the cylinder.

Let M be a point on the surface of the cylinder such that $OM=b$, and draw lines MQ and MP, perpendicular and parallel to Y-axis, respectively. (See Figure 1.) If Φ is the angle between OP and OM, then $OP=b\cos\Phi$ and $OQ=b\sin\Phi$.

Now if the curve drawn by the compass is to be an ellipse with semiaxes a and b, its parametric equations have to be $x = \text{arc } OP = a \cos \beta$ and $y = OQ = b \sin \beta$. But $OQ = b \sin \Phi$, and so we can take Φ to be our parameter and obtain the following equations for the ellipse:

$$x = a \cos \Phi \qquad \text{and} \qquad y = b \sin \Phi,$$

and this implies the following relation for the curve C:

$$\frac{\text{arc } OP}{\text{chord } OP} = \frac{a \cos \Phi}{b \cos \Phi} = \frac{a}{b}. \tag{1}$$

Let the equation of curve C in polar coordinates be $\rho = f(\theta)$, where O is the pole. Then arc $OP/\rho = a/b$ or arc $OP = \rho(a/b)$. Therefore

$$\rho \frac{a}{b} = \int_{\theta_0}^{\theta} \sqrt{\rho^2 + (\rho')^2} d\theta \tag{2}$$

θ_0 being the angle corresponding to $\rho = 0$. The solution of (2) is the logarithmic spiral

$$\rho = \rho_0 \exp\left[\frac{\theta}{\alpha}\right] \tag{3}$$

where

$$\alpha = \sqrt{(a^2/b^2) - 1} = \frac{e}{\sqrt{(1 - e^2)}} \tag{4}$$

and e is the eccentricity of the ellipse.

In practice, since the logarithmic spiral tends asymptotically toward the origin, the useful portion of the curve C, as far as our problem is concerned, may start from the point $A(\rho_0, 0)$ (Figure 2). The portion OA of the curve C can be

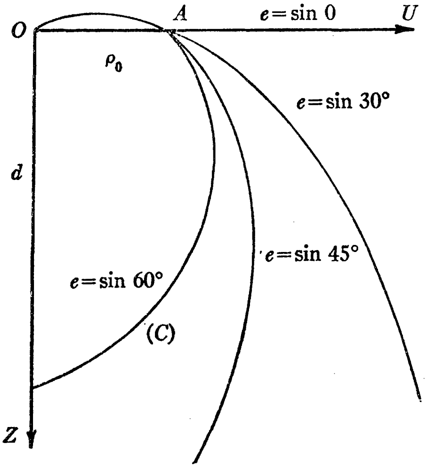

FIG. 2

approximated by an arc of an arbitrary curve such that arc $OA=(a/b)\rho_0$. A mirror symmetry with respect to OZ, or any other line OV in the plane UOZ, gives the complete section of the cylinder. Each curve C corresponds to a given value of e. The value of ρ_0 is arbitrary but it can be taken such that the free end of C is the intersection of the curve with the z axis at a predetermined distance d from O.

d is then the upper limit of b and we have

$$\rho_0 = d \exp\left[-\pi/2\alpha\right]. \tag{5}$$

Figure 2 presents several logarithmic spirals for different values of e. When e tends to zero, the curve C tends to the line OU and the cylinder is a plane as it is expected.

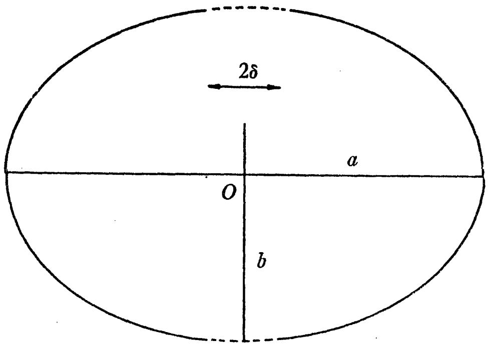

FIG. 3

In drawing the ellipse there is a portion in the middle, of width 2δ (Figure 3), that is only approximate. In terms of d and e we have:

$$\delta = \frac{d}{\sqrt{(1-e^2)}} \exp\left[-\frac{\pi\sqrt{1-e^2}}{2e}\right]. \tag{6}$$

Obviously δ is very small when the ellipse is almost circular

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ON VECTOR REPRESENTATION OF RIGID BODY ROTATION

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The general method of determining the final position of a rigid body after undergoing a finite rotation about some preassigned axis is by computing the Euler angles under a unitary transformation. This may be done also by using quaternions or Cayley-Klein parameters. (See for example [1], Chapter I.) However a precise description of such a rotation may be given by the solution of certain first order vector differential equations. In this note, we present two simple methods of solving such an equation.

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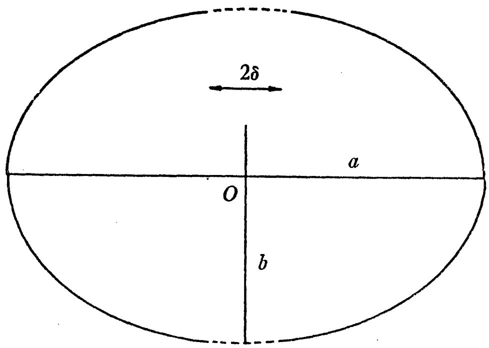

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Let the preassigned axis of rotation be denoted by the unit vector $\mathbf{u}$, and the initial vector by $\mathbf{r}_0$. The rotation of $\mathbf{r}_0$ through an arbitrary angle θ with respect to $\mathbf{u}$ is then described by the solution of the vector differential equation

$$\mathbf{r}'(\theta) = \mathbf{u} \times \mathbf{r}(\theta), \qquad \mathbf{r}(0) = \mathbf{r}_0. \tag{1}$$

In order that (1) describes a genuine rotation, we may assume $\mathbf{u}\times\mathbf{r}_0 \neq 0$. In this case $\mathbf{u}$, $\mathbf{r}_0$, and $\mathbf{u}\times\mathbf{r}_0$ form a linearly independent set of vectors. We may then write the solution of (1) as

$$\mathbf{r}(\theta) = \lambda(\theta)\mathbf{u} + \mu(\theta)\mathbf{r}_0 + \nu(\theta)(\mathbf{u} \times \mathbf{r}_0) \tag{2}$$

where $\lambda(\theta)$, $\mu(\theta)$, and $\nu(\theta)$ are to be determined. Using the given differential equation we have

$$\begin{aligned} \mathbf{r}'(\theta) &= \lambda'(\theta)\mathbf{u} + \mu'(\theta)\mathbf{r}_0 + \nu'(\theta)(\mathbf{u} \times \mathbf{r}_0) \\ &= \nu(\theta)(\mathbf{u}\cdot\mathbf{r}_0)\mathbf{u} - \nu(\theta)\mathbf{r}_0 + \mu(\theta)(\mathbf{u} \times \mathbf{r}_0). \end{aligned}$$

We equate the coefficients of $\mathbf{u}$, $\mathbf{r}_0$ and $\mathbf{u}\times\mathbf{r}_0$ and apply the initial conditions $\mathbf{r}(0)=\mathbf{r}_0$ and $\mathbf{r}(0)=\mathbf{u}\times\mathbf{r}_0$ to obtain

$$\begin{cases} \lambda'(\theta) = \nu(\theta)(\mathbf{u}\cdot\mathbf{r}_0), \\ \mu'(\theta) = -\nu(\theta), \\ \nu'(\theta) = \mu(\theta). \end{cases}$$

with initial conditions $\lambda(0)=\nu(0)=\lambda'(0)=\mu'(0)=0$ and $\mu(0)=\nu'(0)=1$. In particular, we have $\nu''(\theta)+\nu(\theta)=0$. Thus $\nu(\theta)=\sin\theta$, $\mu(\theta)=\cos\theta$, and $\lambda(\theta)=(\mathbf{u}\cdot\mathbf{r}_0)(1-\cos\theta)$. Consequently the solution to (1) is

$$\mathbf{r}(\theta) = (1 - \cos\theta)(\mathbf{u}\cdot\mathbf{r}_0)\mathbf{u} + \cos\theta\,\mathbf{r}_0 + \sin\theta(\mathbf{u} \times \mathbf{r}_0). \tag{3}$$

A simpler way of obtaining (3) is to proceed as follows: Differentiate (1) with respect to θ and obtain

$$\begin{aligned} \mathbf{r}''(\theta) &= \mathbf{u} \times \mathbf{r}'(\theta) = \mathbf{u} \times (\mathbf{u} \times \mathbf{r}(\theta)) \\ &= (\mathbf{u}\cdot\mathbf{r})\mathbf{u} - (\mathbf{u}\cdot\mathbf{u})\mathbf{r}(\theta). \end{aligned}$$

Since $\mathbf{u}\cdot\mathbf{r}(\theta)=\mathbf{u}\cdot\mathbf{r}_0$ and $\mathbf{u}$ is a unit vector, we have

$$\mathbf{r}''(\theta) + \mathbf{r}(\theta) = (\mathbf{u}\cdot\mathbf{r}_0)\mathbf{u}. \tag{4}$$

By the method of variation of parameters, we may now obtain the general solution to (4) as $\mathbf{r}(\theta)=\mathbf{A}\cos\theta+\mathbf{B}\sin\theta+(\mathbf{u}\cdot\mathbf{r}_0)\mathbf{u}$ where $\mathbf{A}$ and $\mathbf{B}$ are vector constants of integration. Using the initial conditions as above, we easily obtain (3).

If we write out equation (3) in its component form, it is easily verified that it agrees with the standard formulae for finite rotation. (See [1] p. 8, or [2] p. 192.)

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SYLVESTER'S PROBLEM ON COLLINEAR POINTS

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1. Introduction. We consider, for fixed $n \geqq 3$, configurations K_n consisting of n noncollinear points in the real projective plane, together with all lines containing at least two of the points. Let $t_2(K_n)$ denote the number of *ordinary lines* of K_n, that is, the number of lines containing exactly two points of K_n. For fixed n, let $t_2(n)$ denote the minimum of $t_2(K_n)$, for all K_n. The original problem posed by Sylvester in 1893 is then:

Show that $t_2(n) \geqq 1$ *for all* n.

Solutions to this problem have been given a number of times, the first one apparently by Gallai in 1933 when Erdös asked him the same question independently. The best result to date is that of Kelly and Moser [**5**], who showed that, in fact, $t_2(n) \geqq 3n/7$, with equality holding if $n=7$. Their paper contains an essentially complete bibliography up to 1958. The bibliographies of the recent papers [**1, 3, 4**] will bring the reader up to 1966. It has been widely conjectured that, in fact, $t_2(n) \geqq n/2$ if $n \neq 7$. (This was originally one of three conjectures about $t_2(n)$ made by Dirac in [**2**]. The falsity of one of the others is shown by Böröczky's examples in Section 3.) Section 2 presents an example, due to McKee, which disproves this general conjecture by showing that $t_2(13) \leqq 6$. However, it still seems reasonable to conjecture that $t_2(n) \geqq n/2$ if $n \neq 7$, 13.

In Section 3 we describe examples due to Böröczky, but apparently not published before, which show that $t_2(n) \leqq k$ if $n=2k$, and $t_2(n) \leqq 3k$ if $n=4k+1$ or $4k+3$. In Section 4 we use these known bounds, together with some auxiliary equations, to calculate $t_2(n)$ for all $n \leqq 13$. It is to be hoped that refinements of these methods might also determine $t_2(n)$ for larger values of n.

2. A 13-point configuration with only six ordinary lines. Let AB be the common edge of two congruent regular pentagons in the Euclidean plane. Let the vertices of the pentagons be $ABCDE$ and $ABC'D'E'$, in cyclic order, respectively. Augment the plane by a line l_∞ "at infinity" to obtain a real projective plane in the usual way. Let M be the midpoint of AB, and I, J, K, L the points on l_∞ in the directions CD, MD, ED, AB respectively. See Figure 1. The only lines containing exactly 2 of these 13 points are the dotted lines AJ, BJ, DL, $D'L$, MI, and MK in Figure 1.

This proves that $t_2(13) \leqq 6$. But from $t_2(n) \geqq 3n/7$ we have $t_2(13) \geqq 39/7$. Hence $t_2(13)=6$.

3. An upper bound for $t_2(n)$. The following examples were invented by Böröczky, and partially rediscovered by McKee (both of them being undergraduates at the time). Let n be even, say $n=2k$ ($k=3, 4, \cdots$), and construct the k vertices of a regular k-gon P in the Euclidean plane. Adjoin a point of l_∞ in each of the k directions determined by pairs of vertices of P. Figure 2 shows this construction for $n=12$. These n points and the lines determined by them form a K_n having exactly $n/2$ ordinary lines, one at each vertex of P.

In case $n=4k+1$ ($k=2, 3, \cdots$) first construct a K_{n-1} from a regular $2k$-gon, as just described. Then adjoin a point C at the center of the $2k$-gon, and con-

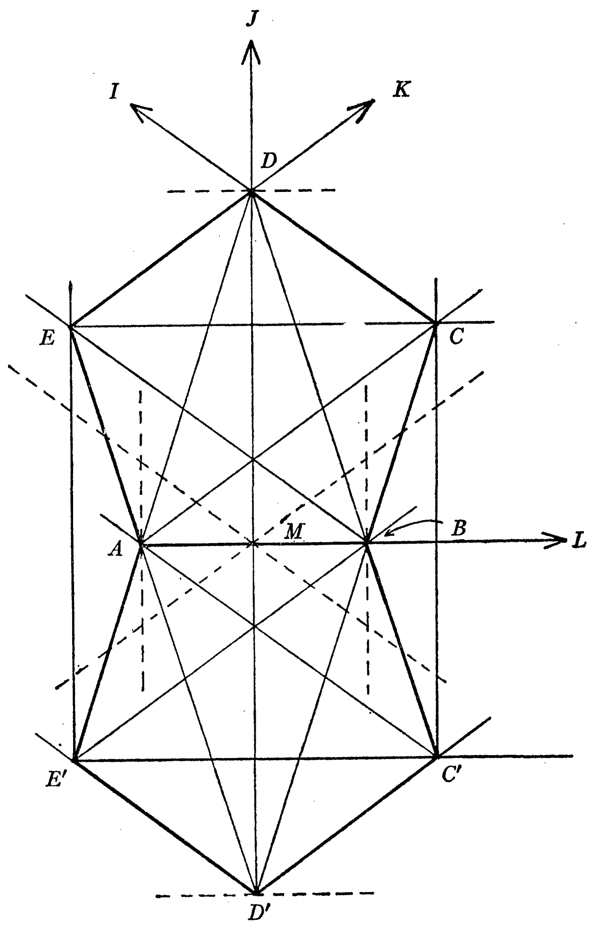

FIG. 1

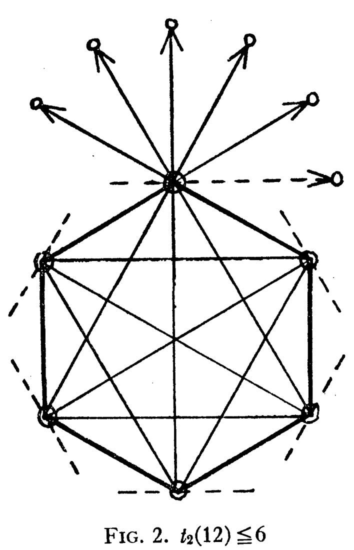

FIG. 2. $t_2(12) \leqq 6$

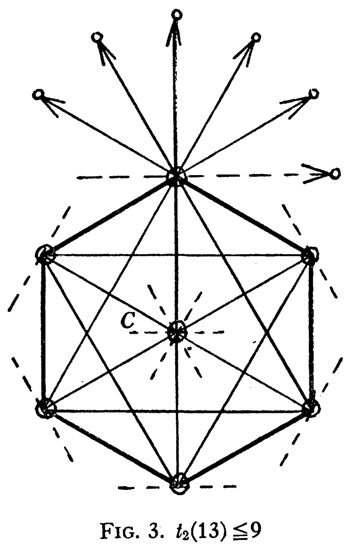

FIG. 3. $t_2(13) \leqq 9$

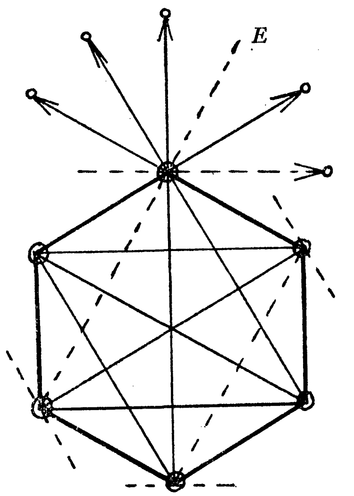

FIG. 4. $t_2(11) \leqq 6$

sider all lines determined by these n points. The resulting configuration has an ordinary line through each of the $2k$ vertices of the $2k$-gon, and k ordinary lines through C, making a total of $3k/4$ ordinary lines. Hence $t_2(4k+1) \leqq 3k/4$. Figure 3 shows the case $n=13$.

In case $n=4k+3$, construct a K_{n+1} from a regular $(2k+2)$-gon, as described for even n. Then delete a point E on l_∞, in a direction *not* determined by an edge of the $(2k+2)$-gon. Consider the configuration consisting of these $4k+3$ points, and all lines determined by them. It has an ordinary line through all but two (opposite) vertices of the $(2k+2)$-gon, as well as k parallel ordinary lines in the direction of E. Hence $t_2(4k+3) \leqq 2k+k=3k$. Figure 4 shows the case $n=11$.

The only known cases where Böröczky's upper bounds are not in fact the *exact* values of $t_2(n)$ are for $n=7$, 13. It is possible that a careful study of these counterexamples might yield further exceptional cases.

4. Calculation of $t_2(n)$ for $n \leqq 13$. The results in the preceding sections determine $t_2(n)$ completely for all $n \leqq 13$, except for $n=5$, 9, 11. Two general lemmas are useful for determining these other three values. The first is just Lemma 4.1 of [**5**].

LEMMA 1. *If t is the total number of distinct lines determined by a set of n points, if exactly $n-r$ of the points lie on some line, and if $n \geqq 3r/2 \geqq 3$, then*

$$t \geqq rn - (3r+2)(r-1)/2. \tag{1}$$

In the sequel we write simply t_i $(i=2, 3, \cdots)$ for the number of lines containing exactly i points of a given configuration.

LEMMA 2. *For any configuration of n points,*

$$\binom{n}{2} = \sum_{i=2} \binom{i}{2} t_i. \tag{2}$$

Proof. The left side of the equation counts the number of (not necessarily distinct) lines determined by n points in pairs. The right side counts the same lines, each term, $\binom{i}{2} t_i$, being the number of (not necessarily distinct) lines which contain exactly i points.

The value of $t_2(5)$ is now readily found. Kelly and Moser's bound yields $t_2(5) \geqq 3$, and the configuration determined by five points, of which exactly four are collinear, shows that $t_2(5) \leqq 4$. Now equation (2), with $n=5$, reads $10 = t_2 + 3t_3 + 6t_4$, and has no integral solution if $t_2 = 3$. Hence $t_2(5) = 4$.

We now show that $t_2(9) = 6$. The Pappus configuration of Figure 5 shows that $t_2(9) \leqq 6$. Since $t_2(9) \geqq 3 \cdot 9/7 > 3$ we need only show that $t_2(9)$ is not 4 or 5. If $t_2(9) = 4$, then equation (2) reduces to $32 = 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 + 28t_8$. This diophantine equation has no solution for nonzero t_8, t_7, or t_6. Hence, in fact, a solution implies $t_5 = 2$. But the only arrangement of nine points on two lines each containing five points gives rise to 16 ordinary lines, contradicting the assumption $t_2(9) = 4$.

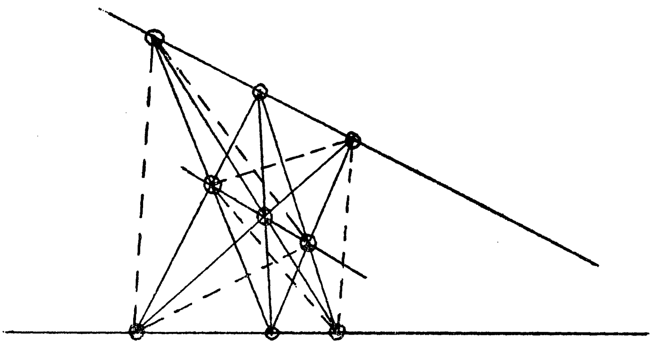

FIG. 5. $t_2(9) \leqq 6$

If $t_2(9) = 5$ then equation (2) becomes $31 = 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7 + 28t_8$. As before, $t_8 = 0$. If $t_7 > 0$, then $t_5 = 1$, which is geometrically impossible with only nine points. Hence $t_7 = 0$. Likewise $t_6 = 0$ and the equation reduces to $31 = 3t_3 + 6t_4 + 10t_5$. Only if $t_5 = 1$ does this equation have a solution. In this case, $t_4 > 1$ is impossible with only nine points. If $t_5 = t_4 = 1$, then two arrangements are possible, but at least nine ordinary lines are formed in each one, contradicting $t_2 = 5$. The only remaining possibility is $t_5 = 1$, $t_3 = 7$, so that the total number of lines is $t_2 + t_3 + t_5 = 13$. But according to Lemma 1 (with $n = 9$, $r = 4$) at least 15 lines are determined. Hence $t_5 \neq 1$, and the diophantine equation has no solutions. Hence $t_2(9) \neq 5$. This completes the proof that $t_2(9) = 6$.

We now show that $t_2(11) = 6$. From the bounds already established, we know that $t_2(11)$ is 5 or 6. We have only to show $t_2(11) \neq 5$. Suppose, if possible, $t_2(11) = 5$, so that equation (2) becomes $50 = 3t_3 + 6t_4 + 10t_5 + 15t_6 + 21t_7$, after we note that there are no solutions if $t_i > 0 (i > 7)$. If $t_7 > 0$, then there is no solution unless $t_5 = 2$, which is geometrically impossible. Hence $t_7 = 0$. If $t_6 > 0$ then $t_6 = 1$ or 2. Each case is possible only if $t_5 = 2$. But eleven points cannot be distributed on three lines each containing at least five of them. Hence $t_6 = 0$, and equation (2) reduces to $50=3t_3+6t_4+10t_5$, which has solutions only if $t_5=2$ or 5. The latter case is clearly impossible. If $t_5=2$ then the two lines, each of which contains five points, either meet in one of the points, or not. Two or one points, respectively, are on neither line. In both cases more than five ordinary lines are formed, thus contradicting $t_2(11)=5$. This completes the proof that $t_2(11)\neq 5$, and hence that $t_2(11)=6$, as required.

These results are tabulated below.

n	3	4	5	6	7	8	9	10	11	12	13
$t_2(n)$	3	3	4	3	3	4	6	5	6	6	6

Since the original preparation of this paper P. D. T. A. Elliott has made further contributions to the problem in [**6**], where he uses the results of [**5**] to prove analogous theorems about circles. Elliott also improves some of the results in [**5**], and repeats the general conjecture that $t_2(n)\geqq n/2$ if $n\neq 7$.

The work reported in this paper was supported by an NSF Undergraduate Research Participation grant to McKee, under the direction of Crowe.

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EUCLID, YOU MUST BE KIDDING

TERRY L. JENKINS, University of Wyoming

Educators in the field of mathematics are more than aware of the thesis in this note. To geometrize or not to geometrize? That is the question. More precisely, to Euclid or not to Euclid is the question. A conclusive victory for the proponents or opponents of teaching Euclidean geometry at the secondary level is not attained in this brief paper. It is hoped however that the reader will find validity in the reasons given for revising, if not replacing entirely, the type of geometry taught. The author's credentials in this area are meager in comparison with the many authoritative accounts written by accomplished mathematicians. Most of my impressions have evolved from visits with many high school teachers while teaching in National Science Foundation and Academic Year Institutes in Nebraska and Wyoming. The participants in the institutes have come from equation (2) reduces to $50=3t_3+6t_4+10t_5$, which has solutions only if $t_5=2$ or 5. The latter case is clearly impossible. If $t_5=2$ then the two lines, each of which contains five points, either meet in one of the points, or not. Two or one points, respectively, are on neither line. In both cases more than five ordinary lines are formed, thus contradicting $t_2(11)=5$. This completes the proof that $t_2(11)\neq 5$, and hence that $t_2(11)=6$, as required.

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n	3	4	5	6	7	8	9	10	11	12	13
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Since the original preparation of this paper P. D. T. A. Elliott has made further contributions to the problem in [6], where he uses the results of [5] to prove analogous theorems about circles. Elliott also improves some of the results in [5], and repeats the general conjecture that $t_2(n)\geqq n/2$ if $n\neq 7$.

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In an informal discussion with a group of teachers it was revealed that what we were doing in the college class was completely new to them although each had taught high school geometry. The comment was not an isolated case. Pursuing this point brought out the fact that a large majority of the teachers had never had a college geometry course prior to their teaching of geometry in high school. A background in algebra, trigonometry, analytic geometry, and calculus was present in varying degrees but essentially a void existed with regard to one of the most exacting mathematical disciplines, one which they surely would be required to teach. Appalling as it may sound, it is not front page news, nor am I considering the problem from this vantage point. The expository literature is saturated with articles complaining about the inadequate preparation of the secondary school teacher. A partial solution to this problem has been attained with the initiation of summer or academic year institutes in the various disciplines. A number of colleges have gone further by requiring fewer courses on "how" to teach and more courses on "what" to teach. This philosophy has, I believe, the support of the majority of teachers. I do not want to enter this ancient arena of battle here but rather I choose to consider a topic taught at the secondary level.

After discussing the problem of high school geometry with many participants in various institutes I carried out a survey of many of the current technicolor texts in the area of high school geometry. In most texts the number of proverbial trees was overwhelming and there was but a hint of the existing forest. There was a myriad of rote exercises, literally colorful in many cases but requiring very little of the so called discovery method. Facts were piled upon facts, most often in haste, in order to build an impressive superstructure. Little time, if any, was spent inspecting the foundation or the materials used. Shortly after matriculation, the college freshman discovers to his dismay that he has gained very little insight into axiomatic systems which is at the foundation of not only geometry but also much of mathematics.

For 2000 years educators have regarded the study of geometry as an integral part of the making of a truly educated person. More realistically, it serves as an introduction to correct logical reasoning which in turn guides the student toward that elusive goal of "mathematical maturity." Ah-h, there is the rub. Is the current presentation of geometry at the secondary level fulfilling its purpose? It is the feeling of the author that it is not. To be sure, this is not to say that what is being taught is purposeless. This would be a harsh, inaccurate, and discouraging generalization. The dissension is founded on the hypothesis that the student is led to believe his geometrical training started with the foundations and then proceeded to its apex, neither of which impression is correct. For the most part, this instruction is carried out under the guise of maturing the student. It is surely to be agreed that the language and content of high school geometry texts is often inaccurate and they at all times assume a very naive posture. This must be a dilemma in all disciplines confronting foundations. The proponents of Euclidean geometry are quick to point out that the students are not ready for the facts of geometry. If the subject is properly approached, this attitude is becoming less a fact and more a defense with each ensuing term. Even if this is so, I do not contend that the facts (foundations) of Euclidean geometry be explored at this level. Furthermore, I do not view the teaching of Euclidean geometry as a sacred obligation of the secondary schools.

At the risk of incurring the wrath of many readers, I would go further to say that Euclidean geometry is not in the mainstream of current mathematics, which is evidenced in the lack of availability of geometers at the college level. Many of the major (and minor) colleges do not require a course in geometry for a mathematics major. This does not lessen the importance or beauty of the subject yet it should vindicate a reappraisal of the curriculum in high schools. There are at present many studies being carried out in this direction whose recommendation hopefully will have far reaching consequences.

The shortcomings of Euclid's original five postulates have been evaluated and the deficiencies have been corrected in the college textbooks. This has necessitated enlarging the list of postulates to approximately twenty-one. A rigorous development through the foundations of Euclidean geometry then poses a formidable task, rarely accomplished unless it be a labor of love. Yet the secondary schools have been assigned the task to do in whole or in part. Considering the assignment, there have been remarkable strides forward and as much was accomplished as could be expected. The strides, however, have been at the expense of rigor, out of necessity, which in turn deprives the student of the deeper insights and consequently the maturity sought. The author maintains that as much insight and more maturity can be gained by studying non-Euclidean geometries and in particular, finite geometries. This brief treatise could not outline the specifics for such a course of study but merely point out a few of its advantages. First, the study of finite geometries can be done thoroughly without assuming a naive posture. Such a study provides the student with more than one model and, most importantly, it gives him the opportunity of exhausting the logical inferences of a few controlled axioms. This can be accomplished over a relatively short time interval for each model. The current Euclidean approach drags through an entire semester, often appearing disjoint and losing the interest of even the capable student. Opponents of the proposed finite geometries may maintain that the abstract models needed in such a study pose as many subtleties as Euclidean geometry, a view with which I do not concur. There are numerous concrete examples of finite geometries which are accessible to the secondary student, examples which vary in the degree of complexity. In some instances, the examples are closely aligned with the elementary aspects of modern algebra.

When, then, is Euclidean geometry to be introduced? A meaningful foundational introduction could be at the end of the proposed secondary level course on finite geometries. After the high school student has gained some insight into the axiomatic techniques he will be more willing to accept the possibility of a larger and more complicated system. This introduction to Euclidean geometry would be meant simply to acquaint the student with the axioms involved and some of the basic foundational proofs. A complete analysis of Euclidean geometry would not even be attempted at this point for two reasons. The first reason is clear. There is an upper bound to the number of courses which can be squeezed into a high school curriculum. Secondly, it is entirely possible that a student is interested in mathematics but not specifically in geometry. It is true that at one time Euclidean geometry was at the core of mathematics. The status of geometry at present is simply that of being one of the many fascinating areas into which an aspiring mathematician can concentrate his efforts. If the incoming freshman in college had the proposed geometry background, the colleges would be more inclined toward the task of teaching Euclidean geometry to those who "wanted" to continue in this direction.

The proposed secondary course creates two immediate problems which are, however, superficial. Textbooks which approach geometry from this point of view are not available but if there were a market for such, there are many capable mathematicians who would supply the needed textbooks. Secondly, the question of the high school teacher's preparation would come up. But again, how many were fully prepared to teach Euclidean geometry? This problem can be disposed of by slanting the college geometry course for education majors more toward finite geometry. It has been my experience that many institute members lacked confidence when introduced to finite geometries. I am convinced that this attitude will disappear when they become more familiar with the flavor and are given more than a brief encounter with the subject.

This short note has undoubtedly been an oversimplification of the problem and a clear cut solution has not been attempted here. Only a partial insight into the problem was intended. There are other, perhaps more deserving, suggestions which I happily acknowledge. The author does hope to renew a spark of interest on the part of the academician, a spark which will further the accomplishments in this area to the benefit of student, teacher, and geometry in general.

RELATION BETWEEN THE BETA AND THE GAMMA FUNCTIONS

AUGUSTINE O. KONNULLY, St. Albert's College, Ernakulam, India

The expression of the beta function in terms of the gamma function is well known. In this note we provide a new proof for this result which is based on the following expression for the gamma function in terms of the beta function:

$$\Gamma(p) = \lim_{Re(q)\to\infty} q^p B(p, q), \; Re(p) > 0, \; Re(q) > 0), \tag{1}$$

(See [1] p. 254, Example 5).

Proof. Let s be the real part of $q-1$ and r, the imaginary part. By the substitution $t = sx$, we have

$$s^p \int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^s t^{p-1}\left(1 - \frac{t}{s}\right)^{q-1} dt.$$

the basic foundational proofs. A complete analysis of Euclidean geometry would not even be attempted at this point for two reasons. The first reason is clear. There is an upper bound to the number of courses which can be squeezed into a high school curriculum. Secondly, it is entirely possible that a student is interested in mathematics but not specifically in geometry. It is true that at one time Euclidean geometry was at the core of mathematics. The status of geometry at present is simply that of being one of the many fascinating areas into which an aspiring mathematician can concentrate his efforts. If the incoming freshman in college had the proposed geometry background, the colleges would be more inclined toward the task of teaching Euclidean geometry to those who "wanted" to continue in this direction.

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$$s^p \int_0^1 x^{p-1}(1-x)^{q-1}dx = \int_0^s t^{p-1}\left(1-\frac{t}{s}\right)^{q-1} dt.$$

Letting s tend to infinity we obtain

$$\lim_{s\to\infty} s^p B(p, q) = \lim_{s\to\infty} \int_0^s t^{p-1}\left(1 - \frac{t}{s}\right)^{s+ir} dt.$$

Now

$$\lim_{s\to\infty} \int_0^s t^{p-1}\left(1 - \frac{t}{s}\right)^{s+ir} dt = \int_0^\infty t^{p-1}e^{-t}dt = \Gamma(p).$$

For

$$\left|\int_0^\infty t^{p-1}e^{-t}dt - \int_0^s t^{p-1}\left(1 - \frac{t}{s}\right)^{s+ir} dt\right|$$
$$\leqq \left|\int_0^s t^{p-1}\left\{e^{-t} - \left(1 - \frac{t}{s}\right)^{s+ir}\right\} dt\right| + \left|\int_s^\infty t^{p-1}e^{-t}dt\right|$$

and when $0 \leqq t \leqq s$, $s > 1$, we have

$$\left|1 - e^t\left(1 - \frac{t}{s}\right)^{s+ir}\right| \leqq \int_0^t \left|\frac{d}{dt} e^t\left(1 - \frac{t}{s}\right)^{s+ir}\right| dt$$
$$= \frac{1}{s}\int_0^t e^t\left(1 - \frac{t}{s}\right)^{s-1} \mid t + ir) \mid dt$$
$$\leqq \frac{e}{s}\int_0^t (t + \mid r \mid)dt = \frac{e}{s}(t^2/2 + \mid r \mid t)$$

since

$$\exp [t]\left(1 - \frac{t}{s}\right)^{s-1} = \exp\left[t + (s - 1) \log\left(1 - \frac{t}{s}\right)\right]$$
$$= \exp [t/s - \cdots] \leqq e.$$

Hence

$$\left|\int_0^s t^{p-1}\left\{e^{-t} - \left(1 - \frac{t}{s}\right)^{s+ir}\right\} dt\right| \leqq \frac{e}{s}\int_0^\infty e^{-t}t^{Re(p)-1}\left(\frac{t^2}{2} + \mid r \mid t\right) dt < \frac{\epsilon}{2},$$

when $s > N = 2e/\epsilon\{\frac{1}{2}\Gamma(Re(p)+2) + |r|\Gamma(Re(p)+1)\}$.

Also, since $\int_0^\infty t^{p-1}e^t dt$ converges, there exists, for arbitrary $\epsilon > 0$, a number $M > N$ such that $\left|\int_0^\infty t^{p-1}e^{-t}dt\right| < \epsilon/2$, when $s > M$. Thus

$$\left|\int_0^\infty t^{p-1}e^{-t}dt - \int_0^s t^{p-1}\left(1 - \frac{t}{s}\right)^{s+ir} dt\right| < \epsilon \quad \text{for all } s > M.$$

It follows that

$$\lim_{s\to\infty} s^p B(p, q) = \lim_{s\to\infty} \int_0^s t^{p-1}\left(1 - \frac{t}{s}\right)^{s+ir} dt = \Gamma(p)$$

and

$$\lim_{s\to\infty} s^p B(p, q) = \lim_{s\to\infty}\left(\frac{s}{s+1+ir}\right)^p q^p B(p, q) = \lim_{Re(q)\to\infty} q^p B(p, q).$$

THEOREM (Euler). $B(p, q)=\Gamma(p)\Gamma(q)/\Gamma(p+q)$.

Proof. Integrating by parts we have $B(p, q+1)=(q/p)B(p+1, q)$ and, adding the integrals, $B(p, q+1)+B(p+1, q)=B(p, q)$, so that we obtain $B(p, q+1) =(q/p+q)B(p, q)$. By the repeated application of this recursion relation, it follows, for n a positive integer, that

$$B(p, q+n) = \frac{q(q+1)\cdots(q+n-1)}{(p+q)(p+q+1)\cdots(p+q+n-1)}\cdot B(p, q)$$

$$\text{and } B(m, n) = \frac{1, 2\cdots(n-1)}{(m+1)(m+2)\cdots(m+n-1)}\cdot B(m, 1)$$

$$\text{where } B(m, 1) = \frac{1}{m}\cdot$$

Hence we have

$$B(p, q) = \frac{B(p, q+n)\cdot B(q, n)}{B(p+q, n)} = \left(\frac{n}{n+q}\right)^p\cdot\frac{(q+n)^p B(p, q+n)\cdot n^q B(q, n)}{n^{p+q}B(p+q, n)}\cdot$$

Letting n tend to infinity we obtain $B(p, q)=\Gamma(p)\cdot\Gamma(q)|\Gamma(p+q)$.

Reference

1. E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Cambridge University Press, New York, 1958.

BOOK REVIEWS

EDITED BY DMITRI THORO, San Jose State College

Materials intended for review should be sent to: Dmitri Thoro, Department of Mathematics, San Jose State College, San Jose, California 95114.

Modern University Algebra. By Marvin Marcus and Henryk Minc. Macmillan, New York, 1966. xii+244 pp. $6.95.

This unique and exciting introduction to some of the basic concepts of modern mathematics is intended for use in a one-semester or two-quarter course at the freshman or sophomore level. It is especially suitable for mathematics majors but should be very valuable to many social science students. It assumes only that the student has completed the usual high school courses in algebra and trigonometry, although the latter is not essential.

and

$$\lim_{s\to\infty} s^p B(p, q) = \lim_{s\to\infty}\left(\frac{s}{s+1+ir}\right)^p q^p B(p, q) = \lim_{Re(q)\to\infty} q^p B(p, q).$$

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No attempt at an axiomatic development is made; rather, the objective is to penetrate significantly into the subject matter covered.

The presentation is very clear, and the book includes a generous supply of very good exercises and an excellent quiz at the end of each section. Quiz answers and solutions to the exercises are provided at the end of the book.

Chapter 1, "Numbers and Sets," introduces mathematical induction, sets, functions, and cardinality.

Chapter 2, "Combinatorial Analysis," presents a fairly complete theory of permutations on a finite set, and the concept of an incidence matrix is introduced. The important elementary properties of matrices are given in this chapter, although a systematic treatment of linear algebra and matrices is left for a later course.

Chapter 3, "Convexity," introduces the concepts of convex sets and convex functions. It is interesting to see the development of the material without the use of calculus. The triangle inequality, Cauchy's inequality, Minkowski's inequality, and the arithmetic-geometric mean inequality are given here.

Chapter 4, "Rings," presents the concept of a ring as a unifying concept for the three similar algebraic structures of integers, complex numbers, and polynomials. It also includes a section on the theory of equations.

The authors have taken the view that an axiomatic approach is inappropriate at this elementary level and that there are too many important, new, and exciting ideas in mathematics to permit a lengthy involvement with axiomatics. They expressed the hope that this book will prepare students for subsequent courses in linear algebra, modern algebra, and some parts of analysis. In this, it seems clear that the authors have succeeded admirably.

T. O. MOORE, University of Florida

BRIEF MENTION

Intermediate Algebra for College Students, 3rd ed. By T. S. Peterson Harper and Row, New York, 1967. viii+383 pp. $7.50.

Intermediate Algebra, 3rd ed. By P. K. Rees and F. W. Sparks. McGraw-Hill, New York, 1964. xii+330 pp. $5.95.

College Algebra, 5th ed. By P. K. Rees and F. W. Sparks. McGraw-Hill, New York, 1967. x+500 pp. $7.50.

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Plane Trigonometry, 5th ed. By F. W. Sparks and P. K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1965. x+230 pp. $6.50; with tables, $6.75.

Plane Trigonometry with Tables, 3rd ed. By G. Fuller. McGraw-Hill, New York, 1966. xii+328 pp. $6.50.

Fundamental Mathematics, 3rd ed. By T. L. Wade and H. E. Taylor. McGraw-Hill, New York, 1967. xviii + 518 pp. $8.50.

Principles of Mathematics. By P. K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1965. xiii+383 pp. $6.95.

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"The book attempts to provide a suitable terminal course for students of the arts and social sciences, stressing the fundamental concepts and applications of mathematics rather than its formal techniques. However, considerations of this kind would also be beneficial for students of the sciences and prospective teachers of secondary school mathematics who are commonly and naively supposed to acquire an understanding of the reasonableness and relevance of mathematics by osmosis during courses devoted largely to memorized and regurgitated techniques."

Introductory Mathematical Analysis, 2nd ed. By E. D. Eaves and R. L. Wilson. Allyn and Bacon, Boston, 1964. vii+496 pp. $8.75.

Calculus and Analytic Geometry. By J. R. Britton, R. B. Kriegh, and L. W. Rutland. W. H. Freeman, San Francisco, 1966. xiii+1069 pp. $12.50.

A condensation and rearrangement of the authors' two-volume *University Mathematics I, II.*

Calculus and Analytic Geometry, 2nd ed. By A. Schwartz. Holt, Rinehart and Winston, New York, 1967. xiv+1008 pp. $12.50.

Technical Mathematics with Calculus, 2nd ed. By H. S. Rice and R. M. Knight. McGraw-Hill, New York, 1966. xviii+908 pp. $8.50.

Modern Geometry, 2nd ed. By C. F. Adler. McGraw-Hill, New York, 1967. xi+302 pp. $8.75.

More Fun with Figures. By J. A. H. Hunter. Dover, New York, 1966. x+116 pp. $1.00 (paper).

Originally published by Oxford University Press under the title *Figurets: More Fun with Figures.*

Number, The Language of Science, 4th ed., rev. and augmented. By. T. Dantzig. Free Press (Macmillan), New York, 1967. ix+340 pp. $2.45 (paper).

Introduction to Finite Mathematics, 2nd ed. By J. G. Kemeny, J. L. Snell, and G. L. Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1966. xiv+465 pp. $8.95.

It is interesting to note that a Russian translation of the first (1957) edition was published in 1965.

Modern Elementary Statistics, 3rd ed. By J. E. Freund. Prentice-Hall, Englewood Cliffs, N. J., 1967. x+432 pp. $9.25.

Introduction to Probability and Statistics, 2nd ed. By B. W. Lindgren and G. W. McElrath. Macmillan, New York, 1966. ix+288 pp.

Introduction to Probability and Statistics, 2nd ed. By W. Mendenhall. Wadsworth, Belmont, Calif., 1967. xiii+393 pp. $8.50.

The Teaching of Secondary Mathematics, 4th ed. By C. H. Butler and F. L Wren. McGraw-Hill, New York, 1965. x+613 pp. $8.50.

The Theory of Arithmetic, 2nd ed. By J. A. Peterson and J. Hashisaki. Wiley, New York, 1967. xiv+337 pp. $7.50.

Art and Geometry. By W. M. Ivins, Jr. Dover, New York, 1964. x+113, pp. $1.00 (paper).

Originally published in 1946 by Harvard University Press.

ANSWERS

A420. Writing $2^{-m}=(5/10)^m=5^m\cdot 10^{-m}$ demonstrates the desired result since 5^m never ends in zero.

A421. With the given restriction on the a_k, we have $\sum a_k^{-1}<\infty$. Thus any sufficiently large number cannot be represented.

A422. Since there are six different letters in Romney, the length of the period in the repeating decimal must be six. The only digital divisor to yield such periods is seven. So O is seven. N may be any digit between one and six. Four is the only one which meets specifications. $4/7=.571428\ 574128\cdots$ Romney $=571428$.

A423. No. Let the number be of the form $a \cdots b$ then $b \cdots a = 2(a \cdots b)$. Now a can be 0, 1, 2, 3, or 4 and corresponding to these values b can be (0), (2, 3), (4, 5), (6, 7) or (8, 9), respectively. By comparing the last digits, none of these are possible.

A424. Suppose p, $p+2$ are twin primes such that $p^2+(p+2)^2=k^2$, with k an integer. Then $2p^2+4p+4=k^2$. This implies that k^2 hence k is even. Let $k=2n$. Then $2p^2+4p+4=4n^2$ or $p^2+2p+2=n^2$. The left hand side is odd since p is odd. The right hand side is even. Therefore, no solution is possible.

(Quickies on page 50)

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

677. *Proposed by Charles R. Wall, University of Tennessee.*

Various states have sales tax rates of 0, 2, 2.25, 2.50, 3, 3.50, 4, and 5 percent. Knowing these rates, what is the smallest amount (before tax) one would have to spend to determine the tax rate:

(1) in a single purchase?

(2) in as many purchases as desired?

678. *Proposed by Charles W. Trigg, San Diego, California.*

From an 8 by 8 checkerboard, the four central squares are removed.

a) Show how to cover the remainder of the board with right trominoes so as to have no fault line, or exactly two fault lines, or three fault lines.

b) Show that no covering with right trominoes can have four fault lines.

A right tromino is a nonrectangular assemblage of three adjoining squares. A fault line has its extremities on the perimeter so that a portion of the configuration may be slid along it in either direction without otherwise disturbing the relative position of its parts.

679. *Proposed by Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania.*

Prove that $2^\alpha < 1+\alpha$ for $0<\alpha<1$.

680. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider a circle C, tangent to t', t'' and to E externally. Show that the locus of the center of C is a circle.

681. *Proposed by Leon Bankoff, Los Angeles, California.*

Find an obtuse triangle that is similar to its orthic triangle.

682. *Proposed by John Beidler, Scranton University, Pennsylvania.*

Let b be a fixed integer, $b>1$, and k be a positive integer. Let n be an integer such that the expansion of $n!$ in the base b has kn digits. Find $\lim_{k\to\infty} nb^{-k}$.

683. *Proposed by Murray S. Klamkin, Ford Scientific Laboratory, Dearborn, Michigan.*

Two triangles have sides $\sqrt{a^2+b^2}$, $\sqrt{b^2+c^2}$, $\sqrt{c^2+a^2}$ and $\sqrt{p^2+q^2}$, $\sqrt{q^2+r^2}$, $\sqrt{r^2+p^2}$, respectively. Which triangle has the greater area if in addition we have $a^2b^2+b^2c^2+c^2a^2=p^2q^2+q^2r^2+r^2p^2$ and $a>p$, $b>q$?

SOLUTIONS

Late Solutions

Leung Gesing, Hong Kong: 653; Dimitrios Vathis, Agios Nicoloios, Chalcis, Greece: 653.

The Whispered Clue

656. [May, 1967] *Proposed by Sidney Kravitz, Dover, New Jersey.*

The editor of a mathematics journal said to his assistant, "I have here a cryptarithm which shows a two digit number being multiplied by itself. You will note that the subproducts are not shown, only the number being squared and the final product."

The assistant said, "I've tried to solve this cryptarithm but the solution is not unique. If you told me whether the number being squared were odd or even I might be able to give you a solution."

Afraid of being overheard, the editor whispers the answer to his assistant. The assistant said, "I was hoping you'd say that. I now know the solution."

Unfortunately, a spy for a rival mathematical journal had planted a "mike" in the room. Although he has not seen the cryptarithm, he has overheard the entire conversation. He is able to solve the cryptarithm. Can you?

Solution by Zalman Usiskin, University of Michigan.

Since the assistant was able to solve the cryptarithm upon being told whether the number squared was odd or even, before asking this question he must have reduced the problem to possible cryptarithms which have n odd solutions and 1 even solution, or n even solutions and 1 odd solution, $n>1$. There are three possible cryptarithms satisfying these conditions:

$$\begin{array}{r} ab \\ \underline{ab} \\ cddb \end{array}$$

(1) satisfied by ab = 35, 46, 65, or 85 (3 odd, 1 even)

$$\begin{array}{r} ef \\ \underline{ef} \\ ghge \end{array}$$

(2) satisfied by ef = 45, 56, 81, or 91 (3 odd, 1 even)

$$\begin{array}{r} jk \\ \underline{jk} \\ lmnj \end{array}$$

(3) satisfied by jk = 42, 48, or 93 (1 odd, 2 even)

Had the editor whispered "even," the assistant would still not know the solution, for he would not be able to choose between cryptarithms (1) and (2). So, the editor must have said "odd," leading to the spy's solution

$$\begin{array}{r} 93 \\ \underline{93} \\ 8649 \end{array}$$

to the original cryptarithm

$$\begin{array}{r} jk \\ \underline{jk} \\ lmnj \end{array}.$$

Also solved by Stanley Rabinowitz, Far Rockaway, New York; E. P. Stark, Plainfield, New Jersey; Charles W. Trigg, San Diego, California; William Wernick, City College of New York; and the proposer.

Maximum Profit

657. [May, 1967] *Proposed by C. Stanley Ogilvy, Hamilton College, New York.*

Ship A is anchored 9 miles out from a point O on a straight shoreline. Ship B is anchored 3 miles out opposite a point 6 miles from O. A boat is to proceed from A to some point on the shore, pick up a passenger, and take him to ship B. It costs the boat owner $1 per mile to run his boat, whether there is a passenger aboard or not. Where should the owner contract to pick up the passenger so that his net profit (from A to shore to B) shall be a maximum? We can assume that the passenger insists on a straight line course from the pickup point to B.

Comment by the proposer.

It is immediately recognized as a variation on a very familiar calculus problem, so the unwary may dive ahead and set it up as such. If they do, the profit function will differ only by a sign in one place from the straight problem of minimizing distance. This sign will be squared away during the process of setting the derivative equal to 0. The result is two "answers": points $4\frac{1}{2}$ and 9 miles from O. But $x=4\frac{1}{2}$ at a point not level on the curve (purely extraneous), and $x=9$ yields a *minimum* profit—this would be the very worst place for him to pick up the passenger! Indeed, there is no point for maximum profit. The farther away from O, on the side opposite to B, that he can con the passenger into meeting him, the better his net profit for the day. There is a limit to his profit, but it is only approached as $x \to -\infty$.

Ten incorrect solutions were received. All made the assumption that profit would be maximum when distance was minimum, which is not true. Perhaps the similarity to a familiar problem was an unfair trap.

A Cyclic Quadrilateral

658. [May, 1967] *Proposed by Kaidy Tan, Fukien Normal College, Fukien, China.*

Construct a cyclic quadrilateral so that each side touches one of four fixed circles.

Solution by Stanley Rabinowitz, Far Rockaway, New York.

There are infinitely many such cyclic quadrilaterals. Let the given circles be called C_1, C_2, C_3, and C_4. Construct any cyclic quadrilateral with sides s_1, s_2, s_3, s_4 in the plane. Then construct line t_i parallel to s_i and tangent to C_i, $i=1, 2, 3, 4$. Then the quadrilateral with sides t_i will be cyclic since the sum of two opposite interior angles will still be 180°, and its sides are tangent to the four given circles.

Also solved by Leon Bankoff, Los Angeles, California; Huseyin Demir, Middle East Technical University, Ankara, Turkey; and Michael Goldberg, Washington, D. C.

Goldberg noted that a solution in terms of a rectangle is always possible.

An Unmeasurable Set

659. [May, 1967] *Proposed by C. J. Mozzochi, University of Connecticut.*

Let (X, σ) be a measurable space with an uncountable number of measurable sets. Let (Y, τ) be a topological space with an uncountable number of open sets. Let f and g be measurable functions from X into Y. Prove or disprove,

$$E = \{x \mid f(x) = g(x)\} \text{ is measurable.}$$

Solution by the proposer.

The following is a counterexample:

$$X = Y \text{ reals}$$

$$\sigma = \{\text{all sets } A \subset X \text{ such that } A \text{ or } (\sim A) \text{ is finite or countably infinite}\}$$

$$\tau = \{X, \phi, E_\alpha = (\alpha, \infty); \alpha \geqq 2\}$$

$$f(x) = \begin{cases} 0 & \text{if } 0 < x < 1 \\ 1 & \text{if } x \in \sim (0, 1) \end{cases}; \qquad g(x) = 0 \quad \text{for all } x.$$

σ is an uncountable σ-algebra. τ is an uncountable topology. f and g are measurable. However, $E=(0, 1)$ is not measurable.

Remark. If Y is the reals with the usual topology and X is any measurable space, then E is measurable. (Cf. Munroe: *Introduction to Measure and Integra-*

tion, Page 151, and Problem a, Page 152.) The proof appears to depend strongly on the order properties of the reals and the density and countability of the rationals. This suggests that it might not be possible to find a significantly "weaker" set of properties to guarantee for a general topological space Y the measurability of E.

Divisors of a Circle

660. [May, 1967] *Proposed by L. J. Upton, Port Credit, Ontario, Canada.*

Four lines in a plane are concurrent at O. The angles between the lines are each 45°. A circle is superimposed on this configuration so that O lies within the circle. (a) Show that the alternate sectors cover one-half of the circle. (b) Show this result without use of the calculus.

Solution by Michael Goldberg, Washington, D. C.

Let the center of the circle be P, and let AC and BD be two chords through O and at right angles to each other. Let $OP=a$, and let θ be the angle between BD and OP. Then,

$$\begin{aligned}
AC/2 &= \sqrt{r^2 - a^2\cos^2\theta},\\
AO &= \sqrt{r^2 - a^2\cos^2\theta} - a\sin\theta,\\
AO^2 &= r^2 - a^2\cos^2\theta + a^2\sin^2\theta - 2a\sin\theta\sqrt{r^2 - a^2\cos^2\theta},\\
CO &= \sqrt{r^2 - a^2\cos^2\theta} + a\sin\theta,\\
CO^2 &= r^2 - a^2\cos^2\theta + a^2\sin^2\theta + 2a\sin\theta\sqrt{r^2 - a^2\cos^2\theta},\\
AO^2 + CO^2 &= 2r^2 - 2a^2\cos^2\theta + 2a^2\sin^2\theta,\\
BO^2 + DO^2 &= 2r^2 - 2a^2\sin^2\theta + 2a^2\cos^2\theta,\\
AO^2 + BO^2 + CO^2 + DO^2 &= 4r^2.
\end{aligned}$$

The last sum is independent of A and θ. Hence, for any rotation of the two chords through the angle ϕ, the area swept out is $4r^2\phi/2$. If $\phi=\pi/4$, the area is $\pi r^2/2$.

We can generalize to $2n$ equally spaced chords. Then, $\phi=\pi/2n$, and the area swept out is $\pi r^2/n$.

Also solved by Robert X. Brennan; and Huseyin Demir, Middle East Technical University, Ankara, Turkey.

A Functional Equation

661. [May, 1967] *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Find all differentiable functions satisfying the functional equation

$$f(xy) = yf(x) + xf(y).$$

Solution by J. Aczel, University of Waterloo, Canada.

The problem asked for all differentiable solutions of

$$f(xy) = xf(y) + yf(x). \tag{1}$$

We will determine all solutions continuous in at least one point (or, for that matter, all solutions which can be majorized or minimized on a set of positive measure by a measurable function):

For $x=0$, (1) gives $f(0)=yf(0)$ for all y, which is possible only if

$$f(0) = 0. \tag{2}$$

For $x\neq 0$, $y\neq 0$, divide (1) by xy:

$$\frac{f(xy)}{xy} = \frac{f(y)}{y} + \frac{f(x)}{x}$$

or, for

$$g(x) = \frac{f(x)}{x} \tag{3}$$

we have

$$g(xy) = g(x) + g(y). \tag{4}$$

Now, all solutions, for $xy\neq 0$ of this well known equation which are continuous in a point or majorizable (minimizable) on a set of positive measure by a measurable function (see, e.g., J. Aczel, *Lectures on Functional Equations and Their Applications*, Academic Press, New York-London, 1966) are of the form

$$g(x) = c \log |x| \qquad (x \neq 0)$$

with some constant c or, with (3),

$$f(x) = cx \log |x| \qquad \text{for } x \neq 0.$$

Together with (2) we have that all solutions of (1) have to be of the form

$$f(x) = \begin{cases} cx \log |x| & \text{for } x \neq 0 \\ 0 & \text{for } x = 0. \end{cases} \tag{5}$$

On the other hand, the functions (5) evidently satisfy the equation (1) whichever constant c might be. So we have proved, that *the general solution, continuous in a point or majorizable (minimizable) on a set of positive measure by a measurable function, of the functional equation* (1) *is given by* (5), *where c is an arbitrary constant.*

We notice that the functions (5) are *continuous everywhere, even in the origin*, as

$$\lim_{x\to 0} cx \log |x| = 0.$$

On the other hand they are *differentiable in all points* $x\neq 0$, *but not differentiable in the origin*, except for the trivial case $c=0, f(x)\equiv 0$, because

$$\frac{f(x) - f(0)}{x} = \frac{cx \log x - 0}{x} = c \log x$$

is not defined in $x=0$ if $c\neq 0$. So strictly speaking, *if differentiability* (i.e., the existence of a finite derivative) *is required also in the origin, then* $f(x)\equiv 0$ *is the only differentiable solution of the equation* (1).

Also solved by Michael Goldberg, Washington, D. C.; James R. Kuttler and Nathan Rubenstein, Johns Hopkins Applied Physics Laboratory (jointly); Kenneth A. Ribet, Brown University; M. Scarowsky, Montreal, Quebec, Canada; and Stephen E. Spielberg, University of Toledo.

A number of solvers found the function $f(x)=kn \ln x$ but failed to note the lack of a derivative at $x=0$. Thus they found a function which satisfied the functional equation which was continuous everywhere but not differentiable everywhere. One incorrect solution was received.

Zeroes of a Derivative

662. [May, 1967] *Proposed by M. B. McNeil, Univeristy of Bristol, England.*

It is well known that between two real zeroes of a polynomial with real coefficients there is located one real zero of its derivative. Consider the more general question: given the zeroes of a polynomial with possibly complex coefficients, what can be said about the zeroes of its derivative?

Solution by Kenneth A. Ribet, Brown University.

One can prove the following analogue of Rolle's Theorem: the zeroes of the derivative of a polynomial lie in the smallest convex polygon which contains the zeroes of the polynomial.

Let $f(z)=C(z-z_1)(z-z_2)\cdots(z-z_n)$. Then

$$\frac{f'(z)}{f(z)}=\sum_{k=1}^{n}\frac{1}{z-z_k}=\sum_{k=1}^{n}\frac{\bar{z}-\bar{z}_k}{(z-z_k)^2}.$$

Now suppose a is a root of $f'(z)$. If $f(a)=0$, there is nothing to prove, but if $f(a)\neq 0$ we can write

$$0=\sum_{k=1}^{n}\frac{a-z_k}{(a-z_k)^2},$$

which implies

$$a=\left[\sum_{k=1}^{n}\frac{1}{(a-z_k)^2}\right]^{-1}\cdot\sum_{k-1}^{n}\frac{z_k}{(a-z_k)^2}=\sum_{k=1}^{n}\lambda_k z_k$$

for numbers $\lambda_k>0$ such that $\sum_k\lambda_k=1$. Hence a lies in any convex polygon which contains z_k. This proof is given in the form of an exercise by Nehari in his book, *Introduction to Complex Analysis*, Page 122.

Solutions or references also given by Huseyin Demir, Middle East Technical University, Ankara, Turkey; R. J. Driscoll, Loyola University, Illinois; Michael Goldberg, Washington, D. C.; and the proposer.

Comment on Problem 637

637. [November, 1966, and May, 1967] *Proposed by Stanley Rabinowitz, Far Rockaway, New York.*

Prove that a triangle is isosceles if and only if it has two equal symmedians.

Comment by the proposer.

The published solution only proves half of the theorem. The fact that a and b can be interchanged without affecting the equality only proves that if $a=b$, then $k_a=k_b$.

Conversely, if $k_a=k_b$, then $k_a^2=k_b^2$ which implies that $(b^2-a^2)[4a^2b^2c^2+2c^4(b^2+a^2)+2c^6+a^2b^2(b^2+a^2)]=0$, which for positive a, b, c implies that $a=b$.

To see the need for this, suppose

$$k_a = a(a+b-2c)$$
$$k_b = b(a+b-2c),$$

then a and b can be interchanged without affecting the equality but k_a can equal k_b even if $a\neq b$, i.e., when $a+b=2c$; because $k_a-k_b=(a-b)(a+b-2c)$.

Comment on Q397

397. [November 1, 1966] Determine

$$\lim_{n\to\infty} \frac{1}{n}\sqrt[n]{((n+1)(n+2)\cdots(n+n))}.$$

[*Submitted by Murray S. Klamkin*]

Comment by S. Spital, California State Polytechnic College, Pomona.

An alternative solution is provided by a generating power series.

Let $a_n=(n+1)(n+2)\cdots(n+n)/n^n$ and consider

$$\sum_{n=0}^{\infty} a_n x^n.$$

From the root and ratio tests,

$$\lim_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} a_n + 1/a_n$$
$$= \lim_{n\to\infty} \frac{(2n+1)(2n+2)}{(n+1)^2(1+1/n)^n}$$
$$= 4/e.$$

Erratum

Q411 should read " . . . two successive odd primes . . ." or ". . . two successive primes greater than two . . ."

THE JOURNAL OF RECREATIONAL MATHEMATICS

The *Journal of Recreational Mathematics* will publish its first issue in January 1968. JRM will be a quarterly, with a subscription price of \$9.00 per year, \$2.25 per single copy and \$1.00 a year added for foreign postage. The Journal will be published by Greenwood Press, Inc., 211 East 43rd Street, New York, New York 10017, from which subscription information can be obtained.

QUICKIES

From time to time this department will publish problems which may be solved by laborious methods, but which with the proper insight may be disposed of with dispatch. Readers are urged to submit their favorite problems of this type, together with the elegant solution and the source, if known.

Q420. Show that 2^{-m} has exactly m digits in its decimal expansion.

[*Submitted by Gerald C. Dodds*]

Q421. It is well known that every positive rational can be written

$$\sum_{k=1}^{n} a_k^{-1}$$

where the a_k are distinct integers. Is the same true if the a_k are restricted to be squares?

[*Submitted by Albert Wilansky*]

Q422. The Michigan Republicans cry Romney, Romney, Romney, . . . The Nixon Republicans retort No Romney, Romney, Romney, . . .

If $N/O = \cdot$R o m n e y R o m n e y R o m n e y . . . , a repeating decimal, and if each letter represents a number from zero to 9, evaluate Romney.

[*Submitted by Bart Park*]

Q423. Can one find a number (to base 10) which doubles itself on reversing its digits?

[*Submitted by Murray S. Klamkin*]

Q424. Prove that two legs of a right triangle cannot have their lengths equal to twin primes.

[*Submitted by John H. Tiner*]

(Answers on pages 41–42)

ACKNOWLEDGEMENT

The Editorial Board acknowledges with thanks the services of the following mathematicians, not members of the Board, who have kindly assisted by evaluating papers submitted for publication in the MATHEMATICS MAGAZINE.

L. W. Beineke, R. E. Bellman, G. E. Bergum, N. Boukidis, E. E. Burniston, W. B. Caton, J. M. Cibulskis, H. M. S. Coxeter, A. Z. Czarnecki, E. D. Davis, R. F. Dickman, M. P. Drazin, L. C. Eggan, H. W. Eves, R. L. Graham, F. Harary, W. J. Harrington, P. Havas, M. E. Henderson, E. Hewitt, L. H. Hook, S. J. Kelly, J. M. Kingston, V. L. Klee, K. Koh, R. R. Korfhage, D. H. Lehmer, D. Loftsgaarden, E. A. Maier, J. A. Marlin, K. O. May, P. A. Nickel, P. Porcelli, C. J. Potratz, J. Rabung, W. E. Ritter, R. T. Rockafellar, J. L. Selfridge, R. A. Struble, D. F. Ullrich, W. R. Utz, F. H. Young, and H. S. Zuckerman.

"The book attempts to provide a suitable terminal course for students of the arts and social sciences, stressing the fundamental concepts and applications of mathematics rather than its formal techniques. However, considerations of this kind would also be beneficial for students of the sciences and prospective teachers of secondary school mathematics who are commonly and naively supposed to acquire an understanding of the reasonableness and relevance of mathematics by osmosis during courses devoted largely to memorized and regurgitated techniques."

Introductory Mathematical Analysis, 2nd ed. By E. D. Eaves and R. L. Wilson. Allyn and Bacon, Boston, 1964. vii+496 pp. $8.75.

Calculus and Analytic Geometry. By J. R. Britton, R. B. Kriegh, and L. W. Rutland. W. H. Freeman, San Francisco, 1966. xiii+1069 pp. $12.50.

A condensation and rearrangement of the authors' two-volume *University Mathematics I, II.*

Calculus and Analytic Geometry, 2nd ed. By A. Schwartz. Holt, Rinehart and Winston, New York, 1967. xiv+1008 pp. $12.50.

Technical Mathematics with Calculus, 2nd ed. By H. S. Rice and R. M. Knight. McGraw-Hill, New York, 1966. xviii+908 pp. $8.50.

Modern Geometry, 2nd ed. By C. F. Adler. McGraw-Hill, New York, 1967. xi+302 pp. $8.75.

More Fun with Figures. By J. A. H. Hunter. Dover, New York, 1966. x+116 pp. $1.00 (paper).

Originally published by Oxford University Press under the title *Figurets: More Fun with Figures.*

Number, The Language of Science, 4th ed., rev. and augmented. By. T. Dantzig. Free Press (Macmillan), New York, 1967. ix+340 pp. $2.45 (paper).

Introduction to Finite Mathematics, 2nd ed. By J. G. Kemeny, J. L. Snell, and G. L. Thompson. Prentice-Hall, Englewood Cliffs, N. J., 1966. xiv+465 pp. $8.95.

It is interesting to note that a Russian translation of the first (1957) edition was published in 1965.

Modern Elementary Statistics, 3rd ed. By J. E. Freund. Prentice-Hall, Englewood Cliffs, N. J., 1967. x+432 pp. $9.25.

Introduction to Probability and Statistics, 2nd ed. By B. W. Lindgren and G. W. McElrath. Macmillan, New York, 1966. ix+288 pp.

Introduction to Probability and Statistics, 2nd ed. By W. Mendenhall. Wadsworth, Belmont, Calif., 1967. xiii+393 pp. $8.50.

The Teaching of Secondary Mathematics, 4th ed. By C. H. Butler and F. L Wren. McGraw-Hill, New York, 1965. x+613 pp. $8.50.

The Theory of Arithmetic, 2nd ed. By J. A. Peterson and J. Hashisaki. Wiley, New York, 1967. xiv+337 pp. $7.50.

Art and Geometry. By W. M. Ivins, Jr. Dover, New York, 1964. x+113, pp. $1.00 (paper).

Originally published in 1946 by Harvard University Press.

ANSWERS

A420. Writing $2^{-m}=(5/10)^m=5^m\cdot 10^{-m}$ demonstrates the desired result since 5^m never ends in zero.

A421. With the given restriction on the a_k, we have $\sum a_k^{-1}<\infty$. Thus any sufficiently large number cannot be represented.

A422. Since there are six different letters in Romney, the length of the period in the repeating decimal must be six. The only digital divisor to yield such periods is seven. So O is seven. N may be any digit between one and six. Four is the only one which meets specifications. $4/7=.571428\ 574128\cdots$ Romney $=571428$.

A423. No. Let the number be of the form $a\cdots b$ then $b\cdots a=2(a\cdots b)$. Now a can be 0, 1, 2, 3, or 4 and corresponding to these values b can be (0), (2, 3), (4, 5), (6, 7) or (8, 9), respectively. By comparing the last digits, none of these are possible.

A424. Suppose p, $p+2$ are twin primes such that $p^2+(p+2)^2=k^2$, with k an integer. Then $2p^2+4p+4=k^2$. This implies that k^2 hence k is even. Let $k=2n$. Then $2p^2+4p+4=4n^2$ or $p^2+2p+2=n^2$. The left hand side is odd since p is odd. The right hand side is even. Therefore, no solution is possible.

(Quickies on page 50)

PROBLEMS AND SOLUTIONS

EDITED BY ROBERT E. HORTON, Los Angeles City College

Readers of this department are invited to submit for solution problems believed to be new that may arise in study, in research, or in extra-academic situations. Proposals should be accompanied by solutions, when available, and by any information that will assist the editor. Ordinarily, problems in well-known textbooks should not be submitted. Solutions should be submitted on separate, signed sheets. Send all communications for this department to Robert E. Horton, Los Angeles City College, 855 North Vermont Avenue, Los Angeles, California 90029.

PROPOSALS

677. *Proposed by Charles R. Wall, University of Tennessee.*

Various states have sales tax rates of 0, 2, 2.25, 2.50, 3, 3.50, 4, and 5 percent. Knowing these rates, what is the smallest amount (before tax) one would have to spend to determine the tax rate:

(1) in a single purchase?

(2) in as many purchases as desired?

678. *Proposed by Charles W. Trigg, San Diego, California.*

From an 8 by 8 checkerboard, the four central squares are removed.

a) Show how to cover the remainder of the board with right trominoes so as to have no fault line, or exactly two fault lines, or three fault lines.

b) Show that no covering with right trominoes can have four fault lines.

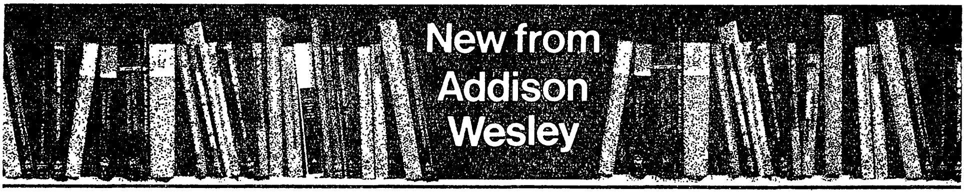
A right tromino is a nonrectangular assemblage of three adjoining squares. A fault line has its extremities on the perimeter so that a portion of the configuration may be slid along it in either direction without otherwise disturbing the relative position of its parts.

679. *Proposed by Gerald C. Dodds, HRB-Singer, Inc., State College, Pennsylvania.*

Prove that $2^\alpha<1+\alpha$ for $0<\alpha<1$.

680. *Proposed by Huseyin Demir, Middle East Technical University, Ankara, Turkey.*

Let E be an ellipse and t', t'' be two variable parallel tangents to it. Consider



Modern Algebra and Trigonometry Second Edition

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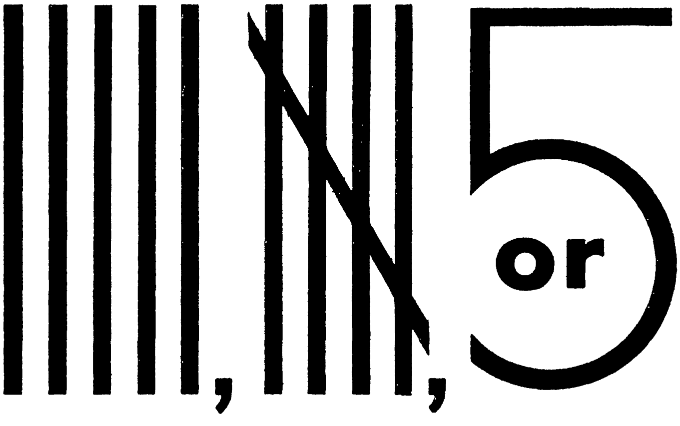

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